An Introduction to Bernstein-Gel'fand-Gel'fand Correspondence

by

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Abstract

This master's thesis serves as an introduction to the Bernstein-Gel'fand-Gel'fand correspondence first written in 1978 [1]. We shall cover some of the fundamental objects that the reader may not be familiar with, mainly tensor algebras and adjunction between categories. Finally we prove the Bernstein-Gel'fand-Gel'fand correspondence for the case of complexes of graded modules as stated in Eisenbud-Fløystad-Schreyer [4], fleshing out all of the details.

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Chapter 1

Introduction

Formally, the Bernstein-Gel'fand-Gel'fand correspondence, shortened to BGG correspondence, is an adjunction between the category of complexes of exterior modules and the category of complexes of modules over a symmetric algebras. An adjunction is a type of categorical equivalence weak enough to be useful but still strong enough to preserve the required structure. This yields a way to port problems and computations from one category to another and back while preserving the structure of each. This is particularly useful since finitely generated modules over the exterior algebra are finite dimensional, which is typically not the case for the symmetric algebra. This provides fertile ground for applications in computational algebra. One of the most well-known applications of the BGG correspondence is an algorithm to compute sheaf cohomology over $\mathbb{P}^n$ developed by David Eisenbud, Gunnar Fløystad, and Frank-Olaf Schreyer [4], already built into Macaulay2. This algorithm was recently extended to any weighted projective stack by Michael K. Brown and Daniel Erman. This is given as Algorithm 6.14 in *Tate resolutions on toric varieties* [2]. At the core of Brown and Erman's work is a generalized BGG correspondence and Brown-Erman's generalization of Eisenbud-Fløystad-Schreyer's theory of Tate resolutions. The next step would be to extend this algorithm to toric stacks. David Eisenbud and Frank-Olaf Schreyer have gotten a lot of mileage out of the BGG correspondence. In two famous algebra papers, *Betti Numbers of Graded Modules and Cohomology of Vector Bundles* [6] and *Resultants and Chow forms via Exterior Syzygies* [5], Eisenbud-Schreyer use the BGG correspondence to draw bridges between seeming disparate algebraic structures. This is all to say that BGG correspondence is an active area of research and a useful tool for anyone interested in computational algebra.

Chapter 2

Preliminaries

Before we discuss the Bernstein-Gel'fand-Gel'fand correspondence we need to discuss the objects that are in correspondence and the setting in which this correspondence takes place. This correspondence is between modules over two seemingly different algebras: the symmetric algebra and exterior algebra. These algebras are quotients of a tensor algebra.

2.1 Algebras

Essentially, an algebra is a ring with a module structure or a module with a ring structure.

Definition. Let R be a ring. An *algebra* over R, also denoted R*-algebra*, is a ring S with a ring homomorphism $\alpha : R \to S$ into the center of S . This defines scalar multiplication given by $\alpha(r)s = rs$. NOTE if S is a commutative ring then any map into S is in its center. This is referred to as a commutative R-algebra.

For our case, we let R be a field k. This implies that our map that defines scalar multiplication is injective. This follows from the facts that fields only have two ideals, (0) and (1), and the kernel is an ideal of k, implying that the kernel is either the whole field or (0). But since $1 \mapsto 1$, the kernel can't be the whole field. Hence we think of a *k-algebra* as a ring that contains a field.

Examples:

- Every ring, with a map into its own center is an algebra over itself.
- Just like every abelian group is a $\mathbb{Z}$-module, any ring is a $\mathbb{Z}$-algebra.

- Polynomial ring $S = R[x_1, ..., x_n]$ in finitely many variables is an algebra over R.
- The complex numbers form a 2-dimensional algebra over the reals.
- The quaternions form a 4-dimensional algebra over the real numbers but not the complex numbers since $\mathbb{C}$ is not in the center of the quaternions.

Definition. A *subalgebra* is a subring S' of S that is an R-module.

Definition. A $homomorphism$ of R-algebras is an R-linear ring homomorphism. Let $\phi : S \to T$ for R-algebras S, T be an R-algebra homomorphism and $x, y \in S$ $r \in R$, then

$$\begin{aligned} \phi(rx) &= r\phi(x), \\ \phi(x+y) &= \phi(x) + \phi(y), \\ \phi(xy) &= \phi(x)\phi(y), \\ \phi(1) &= 1. \end{aligned}$$

2.1.1 Graded Algebras

A *graded ring* is a ring R with a set of additive subgroups $\{R_n\}_{n\geq 0}$ where $R = \bigoplus R_n$ and $R_m R_n \subset R_{m+n}$ for all $m, n \geq 0$. Hence R_0 is a subring of R and each R_n is an R_0-module.

Example. Probably the most familiar graded ring is a polynomial ring. Let $R = k[x_1, ..., x_m]$. Then each graded component R_n is the set of all homogeneous polynomials of degree n. More explicitly, for $\mathbb{R}[x, y]$ we have the following breakdown,

degree	homogeneous components
0	$\mathbb{R}$
1	x, y
2	x^2, xy, y^2
3	x^3, x^2y, xy^2, y^3
4	x^4, x^3y, x^2y^2, xy^3, y^4
$\vdots$	$\vdots$

Definition. Given a graded ring R, a *graded R-module* is an R-module M with a family $(M_N)_{n\geq 0}$ of subgroups of M such that $M = \bigoplus_{n=0}^{\infty} M_n$ and $R_m M_n \subset M_{m+n} \forall m, n \geq 0$. Notice that this makes each M_n an R_0-module. We call an element of M *homogeneous* if that element is in one graded piece. From this we can decompose any element of M into a finite sum of homogeneous components.

A *graded R-algebra* is a graded ring S that is also a graded $R-$module.

2.1.2 Tensor Algebras

Some important and naturally graded algebras are the tensor algebra and its quotient algebras, the exterior algebra and symmetric algebra, that inherit their grading from the original tensor algebra. We define tensor algebra over a field since that is the construction used in the BGG correspondence.

Definition. Let V be a vector space of a field k. We may create a *tensor algebra* of V, denoted $T(V)$, by letting the tensor product define multiplication between elements in our vector space to create an algebra over V. We define the n^{th} tensor power of V to be the n^{th} tensor product of V with itself, denoted $T^n V = V^{\otimes n}$. Then we construct $T(V)$ by taking the direct sum of all the n^{th} tensor powers, $T(V) = \bigoplus_{n \in \mathbb{N}} T^n V$. Notice that $T^i V \otimes T^j V$ and $T^{i+j} V$ are canonically isomorphic. This map determines the multiplication and hence gives $T(V)$ a natural grading by $\mathbb{N}$. We generally refer to this grading by $\mathbb{Z}$ where the subspaces of degree less than zero are 0.

2.1.3 Exterior Algebra

The exterior algebra of a vector space over a field k, denoted $\bigwedge(V)$, is defined to be isomorphic to a quotient algebra of $T(V)$ by the ideal, I, generated by all of the elements of the form $(x \otimes x)$ for all $x \in V$, i.e. $\bigwedge(V) = T(V)/I$. Say $T(V)$ is generated by x, y then $\bigwedge(V) = T(V)/(x \otimes x, y \otimes y)$. We get *skew-commutative* from noticing that $(x+y) \otimes (x+y) = x \otimes x + x \otimes y + y \otimes x + y \otimes y = 0$, since squares go to zero, this implies that $x \otimes y + y \otimes x = 0$.

2.1.4 Symmetric Algebras

Similar to the exterior algebra, the symmetric algebra over a vector space V, denoted $S(V)$, can be constructed as a quotient algebra of $T(V)$. Here the ideal, J, is generated by all of the commutators, $x \otimes y - y \otimes x$. Say V is generated by x, y, z, then $S(V) = T(V)/(x \otimes y - y \otimes x, x \otimes z - z \otimes x, z \otimes y - y \otimes z)$.

Definition. Let M, N be graded $R-$modules, a *homomorphism of graded R-modules* is an R-module homomorphism $f : M \to N$ such that $f(M_n) \subset N_n \forall n \geq 0$, i.e. graded pieces get sent to graded pieces. If these graded pieces get sent to a different degree then we call the map f a *graded map of degree* i, denoted deg(f)=i, such that $f(M_n) \subset N_{n+i} \forall n \in \mathbb{Z}$. Note, when composing graded maps the degrees add.

Example. Given a chain complex, **C** with differential $d_n : C_n \to C_{n-1}$ for all $n \in \mathbb{Z}$, then deg(d)=-1.

Definition. In fact rings, modules, and algebras can be graded by any arbitrary abelian group. An object with a $\mathbb{Z} \times \mathbb{Z}$ grading is called *bigraded.* Similar to before for rings, we have $R = \bigoplus_{(p,q)\in\mathbb{Z}\times\mathbb{Z}} R_{p,q}$ and $R_{i,j}R_{m,n} \subset R_{i+m,j+n}$. The rest follow as above but now with two indices.

Definition. From a bigraded ring, module, or algebra we can create a *bicomplex* or *double complex*: Let M be a bigraded module, denoted $M_{p,q}$ such that $(p, q) \in \mathbb{Z}\times\mathbb{Z}$, with differentials $d^h_{p,q} : M_{p,q} \to M_{p-1,q}$ and $d^v_{p,q} : M_{p,q} \to M_{p,q-1}$ in bidegree (-1,0) and (0,-1) respectively and properties $d^h d^h = 0$, $d^v d^v = 0$, and $d^h_{p,q-1} d^v_{p,q} + d^v_{p-1,q} d^h_{p,q} = 0$ i.e. anticommutativity. Placing each $M_{p,q}$ on lattice points of $\mathbb{Z} \times \mathbb{Z}$ we have the following diagram:

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\dots & M_{p-1,q+1} \arrow[l] \arrow[d, "d^v_{p-1,q+1}"] & M_{p,q+1} \arrow[l, "d^h_{p,q+1}"'] \arrow[d, "d^v_{p,q+1}"] & M_{p+1,q+1} \arrow[l, "d^h_{p+1,q+1}"'] \arrow[d, "d^v_{p+1,q+1}"] & \dots \arrow[l] \\
\dots & M_{p-1,q} \arrow[l] \arrow[d, "d^v_{p-1,q}"] & M_{p,q} \arrow[l, "d^h_{p,q}"'] \arrow[d, "d^v_{p,q}"] & M_{p+1,q} \arrow[l, "d^h_{p+1,q}"'] \arrow[d, "d^y_{p+1,q}"] & \dots \arrow[l] \\
\dots & M_{p+1,q-1} \arrow[l] \arrow[d] & M_{p,q-1} \arrow[l, "d^h_{p,q-1}"'] \arrow[d] & M_{p+1,q-1} \arrow[l, "d^h_{p+1,q-1}"'] \arrow[d] & \dots \arrow[l] \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

Given a commutative diagram, we may turn it into a bicomplex using a sign change to get anti-commutivity. Given $d^h_{p,q-1}d^v_{p,q} - d^v_{p-1,q}d^h_{p,q} = 0$ from the commutivite diagram we define $b^v_{p,q} = (-1)^p d^v_{p,q}$ to get anti-commutivity,

$$
\begin{aligned}
d^h_{p,q-1}b^v_{p,q} + d^v_{p-1,q}b^h_{p,q} &= (-1)^p d^h_{p,q-1}d^v_{p,q} + (-1)^{p-1} d^v_{p-1,q}d^h_{p,q} \\
&= (-1)^p (d^h_{p,q-1}d^v_{p,q} - d^v_{p-1,q}d^h_{p,q}) = 0.
\end{aligned}
$$

Definition. Total Complex: Given a bicomplex M, the total complex of M, denoted Tot(M), is the complex with terms and differentials defined as,

$$
\begin{gathered}
\text{Tot}(M)_n = \textstyle\bigoplus_{p+q=n} M_{p,q}, \\
D_n : \text{Tot}(M)_n \to \text{Tot}(M)_{n-1} \text{ such that,} \\
D_n = \Sigma_{p+q=n}(d^h_{p,q} + d^v_{p,q})
\end{gathered}
$$

Notice that since $\text{Tot}(M)_n$ is the sum of all the modules in total degree $n = p + q$, this corresponds to the diagonal line $y = -x + n$ for an integer n in the lattice.

Lemma. The total complex is a complex.

Proof. First, notice that im(D_n) $\subset \text{Tot}(M)_{n-1}$ since im($d^h_{p,q}$) $\subset M_{p-1,q}$ and im($d^v_{p,q}$) $\subset M_{p,q-1}$ implying that im $(D_n) =$ im $(\Sigma_{p+q=n}(d^h_{p,q} + d^v_{p,q})) \subset \text{Tot}(M)_{n-1}$. Second we need to show that D is a differential, meaning $DD = 0$. This is where anti-commutativity of the bicomplex comes in,

$$\begin{aligned} DD &= \Sigma(d^h + d^v)(d^h + d^v) \\ &= \Sigma(d^h d^h) + \Sigma(d^h d^v + d^v d^h) + \Sigma(d^v d^v) \\ &= \Sigma(d^h d^v - d^h d^v) = 0. \end{aligned}$$

□

2.2 Category Theory

Category theory is often scoffed at as "abstract nonsense" but for much of algebra and mathematics it is a natural setting to talk about more general structure. The language of category theory allows us to compare one collection of objects to another collection of objects.

Definition. A category, $\mathcal{C}$, contains three bits of data:
1. a collection of *objects* denoted obj($\mathcal{C}$),
2. a collection of *morphisms*, denoted Hom(A, B) for every ordered pair of objects,
3. a binary operation called the law of *composition* Hom(A, B) $\times$ Hom(B, C) $\to$ Hom(A, C) denoted $(f, g) \mapsto gf$, for every ordered triple A, B, C of objects.

The above must satisfy the following:
1. There exists an *identity morphism* for each object, $1_A \in$ Hom(A, A) such that $f1_A = f$ and $1_B f = f$ for all $f : A \to B$.
2. Associativity of composition: given morphisms $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, then $(hg)f = h(gf)$.

Examples:

- **Set**: The objects in this category are sets (not proper classes), morphisms are functions, and composition is the usual composition of functions.
- **Groups**: The objects are groups and the morphisms are homomorphims, and composition is the usual composition.

- **Top**: Objects are all topological spaces and morphisms are continuous functions. Note that identity functions are continuous and the composition of continuous functions is continuous.

- **Ab**: Objects are abelian groups and morphisms are homomorphisms.

- **Rings**: Objects are rings and morphisms are ring homomorphisms. Note that we are assuming rings with a unit element and that $\phi(1) = 1$ for every ring homomorphism.

- **ComRings**: Objects are commutative rings and morphisms are ring homomorphisms.

- **Mod**$_R$: Objects are right R-modules for a given ring R, denoted $_R$**Mod** for left R-Modules and morphisms are all R-homomorphims.

- **Vect**$_k$: Objects are k-vector spaces and morphism are k-linear maps.

- **Opposite Category**: Let $\mathcal{C}$ be a category. $\mathcal{C}^{op}$ has the same objects as $\mathcal{C}$, but the morphism in $\mathcal{C}$, $f : X \to Y$, has the domain and codomain swapped, $f^{op} : Y \to X$ and composition of morphisms $gf \in \mathcal{C}^{op}$ is defined to be the composition fg in $\mathcal{C}$.

Definition. A category $\mathcal{S}$ is a *subcategory* of a category $\mathcal{C}$ if the following are satisfied,

(i) obj($\mathcal{S}$) $\subset$ obj($\mathcal{C}$),

(ii) $\text{Hom}_{\mathcal{S}}(A, B) \subset \text{Hom}_{\mathcal{C}}(A, B)$ $\forall A, B, \in$ obj$(\mathcal{S})$, where the Hom sets in $\mathcal{S}$ are denoted by $\text{Hom}_{\mathcal{S}}(-, -)$,

(iii) if $f \in \text{Hom}_{\mathcal{S}}(A, B)$ and $g \in \text{Hom}_{\mathcal{S}}(B, C)$ then the composite $gf \in \text{Hom}_{\mathcal{S}}(A, C)$ is equal to the composite $gf \in \text{Hom}_{\mathcal{C}}(A, C)$,

(iv) if $A \in$ obj$(\mathcal{S})$, then the identity $1_A \in \text{Hom}_{\mathcal{S}}(A, A)$ is equal to the identity $1_A \in \text{Hom}_{\mathcal{C}}(A, A)$.

Definition. A subcategory $\mathcal{S}$ of $\mathcal{C}$ is a *full subcategory* if, $\forall A, B \in$ obj$(\mathcal{S})$, we have $\text{Hom}_{\mathcal{S}}(A, B) = \text{Hom}_{\mathcal{C}}(A, B)$.

2.2.1 Functors

Definition. Functor (covariant): Let $\mathcal{C}$ and $\mathcal{D}$ be categories, a *covariant functor* $\mathbb{F} : \mathcal{C} \to \mathcal{D}$ is a function such that,

(i) if $A \in \text{obj}(\mathcal{C})$, then $\mathbb{F}(A) \in \text{obj}(\mathcal{D})$,

(ii) if $f : A \to A'$ in $\mathcal{C}$, then $\mathbb{F}(f) : \mathbb{F}(A) \to \mathbb{F}(A')$ in $\mathcal{D}$,

(iii) if $A \xrightarrow{f} A' \xrightarrow{g} A''$ in $\mathcal{C}$, then $\mathbb{F}(A) \xrightarrow{\mathbb{F}(f)} \mathbb{F}(A') \xrightarrow{\mathbb{F}(g)} \mathbb{F}(A'')$ in $\mathcal{D}$ and $\mathbb{F}(gf) = \mathbb{F}(g)\mathbb{F}(f)$,

(iv) $\mathbb{F}(1_A) = 1_{\mathbb{F}(A)}\ \forall A \in \text{obj}(\mathcal{C})$.

Example. Forgetful Functor: $\mathbb{F}$: **Groups** $\to$ **Sets**. For a given group G, $\mathbb{F}(G)$ is just the underlying set of G and $\mathbb{F}(f)$ is a homomorphism regarded as a function. Since a group is a set G and an operation $\phi : G \times G \to G$, then $\mathbb{F} : (G, \phi) \mapsto G$. More generally this is what forgetful functors do, they "forget" the structure/s. We have the Forgetful functor $\mathbb{F}$: **Rings** $\to$ **Ab** for some Ring the ordered triple $(R, \phi, \psi) \mapsto (R, \phi)$ where the "multiplication" operation is "forgotten". Similarly for $\mathbb{F}$: **Rings** $\to$ **Sets**.

Example. Free Functor: An example of a free functor is a functor from the category of **Set** $\to$ **Groups**, where a set gets sent to the Free Group generated by that set.

Definition. A *contravariant functor* is a covariant functor $\mathbb{F} : \mathcal{C}^{\text{op}} \to \mathcal{D}$ or $\mathbb{F} : \mathcal{C} \to \mathcal{D}^{\text{op}}$. Meaning the range and domain of morphism are swapped and thus so is composition.

Example. The **Hom Functor** is a map $\mathbf{Hom}(-,-) : \mathcal{C} \to$ **Set**. There is a contravariant and covariant version of the **Hom** fuctor. In the first component we have $\mathbf{Hom}(-, B)$ such that for each $C \in \mathcal{C}$, $C \mapsto \mathbf{Hom}(C, B)$ and for each morphism $f : X \to Y$, $\mathbf{Hom}(B, f) : \mathbf{Hom}(Y, B) \to \mathbf{Hom}(X, B)$. In the second component we have $\mathbf{Hom}(A, -)$ such that for each $C \in \mathcal{C}$, $C \mapsto \mathbf{Hom}(A, C)$ and for each morphism $f : X \to Y$, $\mathbf{Hom}(A, f) : \mathbf{Hom}(A, X) \to \mathbf{Hom}(A, Y)$.

Theorem. Functors preserve isomorphisms of objects. Given $\mathbb{F} : \mathcal{C} \to \mathcal{D}$ and $f : X \to Y$ is an isomorphism in $\mathcal{C}$, then $\mathbb{F}(f)$ is an isomorphism in $\mathcal{D}$.

Proof. Let $g : Y \to X$ such that $fg = 1_Y$ and $gf = 1_X$. Since functors preserve composition and identities, $1_{\mathbb{F}(Y)} = \mathbb{F}(1_Y) = \mathbb{F}(fg) = \mathbb{F}(f)\mathbb{F}(g)$. Similarly for $1_{\mathbb{F}(1_X)}$. □

Definition. Given two functors $\mathbb{F}, \mathbb{G} : \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\tau : \mathbb{F} \to \mathbb{G}$ is a function which assigns to each object $A \in \mathcal{C}$ an map $\tau_A = \tau A : \mathbb{F}(A) \to \mathbb{G}(A)$ of $\mathcal{D}$ in such a way that every map $f : A \to A'$ in C yields a diagram below that commutes.

$$\begin{array}{ccc} A & \quad & \mathbb{F}(A) \xrightarrow{\tau_A} \mathbb{G}(A) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathbb{F}(f)} \qquad\quad \downarrow{\scriptstyle \mathbb{G}(f)} \\ A' & & \mathbb{F}(A') \xrightarrow{\tau_{A'}} \mathbb{G}(A') \end{array}$$

When the diagram commutes we say that $\tau_A : \mathbb{F}(A) \to \mathbb{G}(A)$ is natural. We often call natural transformations a "morphism of functors". In other words a natural transformation is a collection of maps from one diagram to another such that the diagrams commute.

Definition. A *natural isomorphism* or *natural equivalence* is a natural transformation τ where every component τ_A is invertible in $\mathcal{D}$.

2.2.2 Types of Equivalences

Definition. The first and strongest type of equivalence is an *isomorphism of categories*. It is generally too strong to be useful. Given $\mathbb{F} : \mathcal{C} \to \mathcal{D}$ and $\mathbb{G} : \mathcal{D} \to \mathcal{C}$ such that $\mathbb{FG} = \mathrm{Id}_{\mathcal{D}}$ and $\mathbb{GF} = \mathrm{Id}_{\mathcal{C}}$ are the identity functor for their respective categories. This means that both the objects and the morphisms of $\mathcal{C}, \mathcal{D}$ are in one-to-one correspondece. Implying that these categories share all the same categorical properties.

Definition. An *Equivalence of Categories* between two categories $\mathcal{C}, \mathcal{D}$ is a pair of functors $\mathbb{F} : \mathcal{C} \to \mathcal{D}$ and $\mathbb{G} : \mathcal{D} \to \mathcal{C}$ that are natural isomorphisms, $\mathbb{FG} \cong \mathrm{Id}_{\mathcal{D}}$ and $\mathbb{GF} \cong \mathrm{Id}_{\mathcal{C}}$.

Example. A stronger version of the BGG correspondence presented in this paper is an equivalence from the category of graded E-modules to the category of linear free complexes of S-modules. There is a nice treatment of this in chapter 7 of David Eisenbud's The Geometry of Syzygies [3].

Definition. An *Adjunction* is a pair of functors $\mathbb{L} : \mathcal{C} \to \mathcal{D}$, $\mathbb{R} : \mathcal{D} \to \mathcal{C}$ such that for all objects $C \in \mathcal{C}$, $D \in \mathcal{D}$ there is an isomorphism $\mathrm{Hom}_{\mathcal{D}}(\mathbb{L}(C), D) \cong \mathrm{Hom}_{\mathcal{C}}(C, \mathbb{R}(D))$ that is natural in both $\mathcal{C}$ and $\mathcal{D}$. Here naturality means that for all morphims $f : C_1 \to C_2$ in $\mathcal{C}$ and $g : D_1 \to D_2$ in $\mathcal{D}$ we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(\mathbb{L}(C_2), D_1) \arrow[r, "\cong"] \arrow[d, "\mathrm{Hom}_{\mathcal{D}}(\mathbb{L}(f),g)"'] & \mathrm{Hom}_{\mathcal{C}}(C_2, \mathbb{R}(D_1)) \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(f,\mathbb{R}(g))"] \\
\mathrm{Hom}_{\mathcal{D}}(\mathbb{L}(C_1), D_2) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathcal{C}}(C_1, \mathbb{R}(D_2)).
\end{tikzcd}$$

Example. Any student who has taken linear algebra has unknowningly seen an example of an adjunction, the Free-Forgetful adjunction. In linear algebra this was taught as every vector space has a basis and any linear independent set generates a vector space. Making this more explicit, let $\mathbb{R} : \mathbf{Vect}_k \to \mathbf{Set}$ be the forgetful functor and $\mathbb{L} : \mathbf{Set} \to \mathbf{Vect}_k$ be the free functor, where a set is sent to a formal linear combination and each function extends to a unique linear transformation. We say that the forgetful functor is right adjoint to the free functor. This familiar construction can be generalized to any of the familiar categories that has a forgetful functor.

Example. One of the most useful adjunction in algebra is the Hom-Tensor adjunction for graded modules. This adjunction will play a key role in the proof of the BGG correspondence. Given graded a right R-module A_R, bimodule ${}_RB_S$, and right S-module C_S, for graded rings R and S, there is a natural isomorphism given by,

$$T : \mathrm{Hom}_S(A \otimes_R B, C) \to \mathrm{Hom}_R(A, \underline{\mathrm{Hom}}_S(B, C)),$$

where $T(f)(a)(b) = f(a \otimes b)$ and $T^{-1}(g)(a \otimes b) = g(a)(b)$. Notice the underline under $\underline{\mathrm{Hom}}_S(B, C)$. This is called an *internal Hom*, meaning it is the set of all graded maps of degree 0.

Chapter 3

Bernstein-Gel'fand-Gel'fand Correspondence

Fix a field k. Let V, W be finite dimensional dual vector spaces over k. We define the elements of W to have degree 1, so that the elements of V have degree -1. Let V have basis $\{e_1, ..., e_n\}$ and W have basis $\{w_1, ..., w_n\}$. Then we define $E = \bigwedge V$ to be the exterior algebra of V and $W = \text{Sym}(W)$ to be the symmetric algebra of W, graded such that $S_i = \text{Sym}_i(W)$ has degree i and $E_j = \bigwedge^j V$ has degree $-j$. Let M be a graded S-module and N be a graded E-module. Let $\mathbf{M}$ be a complex of graded S-modules and $\mathbf{N}$ a complex of graded E-modules as below. Note that we start with cohomological indexing and then add in the homological indexing later. We start with the following complexes,

$$\mathbf{M} : \cdots \to M^{i-1} \xrightarrow{\partial_M^{i-1}} M^i \xrightarrow{\partial_M^i} M^{i+1} \to \ldots,$$

$$\mathbf{N} : \cdots \to N^{i-1} \xrightarrow{\partial_N^{i-1}} N^i \xrightarrow{\partial_N^i} N^{i+1} \to \ldots.$$

Fixing i for M^i and N^i we define $\mathbf{R}$ and $\mathbf{L}$ to be the following complexes,

$$\mathbf{R}(M^i) : \ldots \to \underline{\text{Hom}}_k(E(j), M^i_j) \xrightarrow{\phi^i_j}, \underline{\text{Hom}}_k(E(j+1), M^i_{j+1}) \to \ldots$$

$$\text{with } \phi^i_j : \alpha \mapsto [e \mapsto \Sigma_n x_n \alpha(e_n e)].$$

$$\mathbf{L}(N^i) : \cdots \to S(j) \otimes_k N^i_{-j} \xrightarrow{\psi_j} S(j+1) \otimes N^i_{-j-1} \to \ldots,$$

$$\text{with } \psi : s \otimes n \mapsto \Sigma_m x_m s \otimes e_m n.$$

Applying $\mathbf{R}$ to each M^i results in the following complex,

$$\cdots \to \mathbf{R}(M^{i-1}) \xrightarrow{F^{i-1}} \mathbf{R}(M^i) \xrightarrow{F^i} \mathbf{R}(M^{i+1}) \to \ldots,$$

$$\text{with } F^i : \alpha(-) \mapsto \partial^i_M \circ \alpha(-).$$

Expanding this out, yields our double complex.

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\ldots \arrow[r] & \underline{\mathrm{Hom}}_k(E(j+1), M^{i-1}_{j+1}) \arrow[r, "F^{i-1}_{j+1}"'] \arrow[u] & \underline{\mathrm{Hom}}_k(E(j+1), M^{i}_{j+1}) \arrow[r, "F^{i}_{j+1}"'] \arrow[u] & \underline{\mathrm{Hom}}_k(E(j+1), M^{i+1}_{j+1}) \arrow[r] \arrow[u] & \ldots \\
\ldots \arrow[r] & \underline{\mathrm{Hom}}_k(E(j), M^{i-1}_{j}) \arrow[r, "F^{i-1}_{j}"'] \arrow[u, "\phi^{i-1}_j"'] & \underline{\mathrm{Hom}}_k(E(j), M^{i}_{j}) \arrow[r, "F^{i}_{j}"'] \arrow[u, "\phi^{i}_j"'] & \underline{\mathrm{Hom}}_k(E(j), M^{i+1}_{j}) \arrow[r] \arrow[u, "\phi^{i+1}_j"'] & \ldots \\
\ldots \arrow[r] & \underline{\mathrm{Hom}}_k(E(j-1), M^{i-1}_{j-1}) \arrow[r, "F^{i-1}_{j-1}"'] \arrow[u, "\phi^{i-1}_{j-1}"'] & \underline{\mathrm{Hom}}_k(E(j-1), M^{i}_{j-1}) \arrow[r, "F^{i}_{j-1}"'] \arrow[u, "\phi^{i}_{j-1}"'] & \underline{\mathrm{Hom}}_k(E(j-1), M^{i+1}_{j-1}) \arrow[r] \arrow[u, "\phi^{i+1}_{j-1}"'] & \ldots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Notice that the arrows are going in opposite direction as in the earlier definition. In this case, the line in the lattice where we gather the modules to create the total complex doesn't change. Now for the sign change, we chose the vertical differential to carry the -1 and write $(-1)^i\phi^i_j$ in place of ϕ^i_j. Letting i, j vary, we have the following total complex,

$$\ldots \xrightarrow{\partial^{t-2}_{\mathbf{R}}} \bigoplus_{i+j=t-1} \mathrm{Hom}_k(E(j), M^i_j) \xrightarrow{\partial^{t-1}_{\mathbf{R}}} \bigoplus_{i+j=t} \mathrm{Hom}_k(E(j), M^i_j) \xrightarrow{\partial^{t}_{\mathbf{R}}} \ldots,$$

$$\mathbf{R}(\mathbf{M})^t = \bigoplus_{i+j=t} \mathrm{Hom}_k(E(j), M^i_j),$$

$$\partial^t_{\mathbf{R}} = \{F^i_j + (-1)^i \phi^i_j\}_{i+j=t}.$$

We define $\mathbf{L}(\mathbf{N})$ in a similar manner. Take each N^i in $\mathbf{N}$ and apply $\mathbf{L}$ to each producing the following double complex:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\dots \arrow[r] & S(j+1)\otimes_k N^{i-1}_{-j-1} \arrow[u] \arrow[r, "G^{i-1}_{j+1}"'] & S(j+1)\otimes_k N^{i}_{-j-1} \arrow[u] \arrow[r, "G^{i}_{j+1}"'] & S(j+1)\otimes_k N^{i+1}_{-j-1} \arrow[u] \arrow[r] & \dots \\
\dots \arrow[r] & S(j)\otimes_k N^{i-1}_{-j} \arrow[u, "\psi^{i-1}_j"'] \arrow[r, "G^{i-1}_{j}"'] & S(j)\otimes_k N^{i}_{-j} \arrow[u, "\psi^{i}_j"'] \arrow[r, "G^{i}_{j}"'] & S(j)\otimes_k N^{i+1}_{-j} \arrow[u, "\psi^{i+1}_j"'] \arrow[r] & \dots \\
\dots \arrow[r] & S(j-1)\otimes_k N^{i}_{-j+1} \arrow[u, "\psi^{i-1}_{j-1}"'] \arrow[r, "G^{i-1}_{j-1}"'] & S(j-1)\otimes_k N^{i-1}_{-j+1} \arrow[u, "\psi^{i}_{j-1}"'] \arrow[r, "G^{i}_{j-1}"'] & S(j-1)\otimes_k N^{i+1}_{-j+1} \arrow[u, "\psi^{i+1}_{j-1}"'] \arrow[r] & \dots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Then we totalize the above double complex to get the following total complex,

$$\dots \to \bigoplus_{i+j=n-1} S(j)\otimes_k N^i_{-j} \xrightarrow{\partial^{t-1}_{\mathbb{L}}} \bigoplus_{i+j=t} S(j)\otimes_k N^i_{-j} \to \dots,$$

$$\mathbf{L}(\mathbf{N})^t = \bigoplus_{i+j=t} S(j)\otimes_k N^i_{-j},$$

$$\partial^t_{\mathbb{L}} = \{G^i_j + (-1)^{i+1}\psi\}_{i+j=t}.$$

Theorem. Bernstein-Gel'fand-Gel'fand (1978): The functor $\mathbf{L}$, from the category of complexes of graded E-modules to the category of complexes of graded S-modules, is a left adjoint to functor $\mathbf{R}$. [4]

Proof. We want to show that $\mathrm{Hom}_{\mathrm{Com(S)}}(\mathbf{L}(\mathbf{N}), M) \cong \mathrm{Hom}_{\mathrm{Com(E)}}(N, \mathbf{R}(\mathbf{M}))$. Below we define $\mathcal{L} : \mathrm{Hom}_{\mathrm{Com(S)}}(\mathbf{L}(\mathbf{N}), M) \to \mathrm{Hom}_{\mathrm{Com(E)}}(N, \mathbf{R}(\mathbf{M}))$, by looking at a chain map as a product of maps in the category of modules. Then we need to check that this product of maps is a chain map.

$$\mathrm{Hom}_{Com(S)}(\mathbf{L}(\mathbf{N}), \mathbf{M}) \subset \prod_{t\in\mathbb{Z}} \mathrm{Hom}_S(\bigoplus_{i+j=t} S(j)\otimes_k N^i_{-j}, M^t)$$

$$(1) \cong \prod_{t\in\mathbb{Z}} \prod_{i+j=t} \mathrm{Hom}_S(S(j)\otimes_k N^i_{-j}, M^t)$$

$$(2) \cong \prod_{t\in\mathbb{Z}} \prod_{i+j=t} \mathrm{Hom}_k(N^i_{-j}, \underline{\mathrm{Hom}}_S(S(j), M^t))$$

$$
\begin{aligned}
(3) &\cong \prod_{t\in\mathbb{Z}} \prod_{i+j=t} \mathrm{Hom}_k(N^i_{-j}, M^t(-j)) \\
&\cong \prod_{t\in\mathbb{Z}} \prod_{i+j=t} \mathrm{Hom}_k(N^i(-j), M^t_{-j}) \\
(4) &\cong \prod_{t\in\mathbb{Z}} \prod_{i+j=t} \mathrm{Hom}_k(N^i \otimes_E E(-j), M^t_{-j}) \\
(5) &\cong \prod_{t\in\mathbb{Z}} \prod_{i+j=t} \mathrm{Hom}_E(N^i, \underline{\mathrm{Hom}}_k(E(-j), M^t_{-j})) \\
(*)\text{re-index} &\cong \prod_{i\in\mathbb{Z}} \prod_{i=t-j} \mathrm{Hom}_E(N^i, \underline{\mathrm{Hom}}_k(E(-j), M^t_{-j})) \\
(6) &\cong \prod_{i\in\mathbb{Z}} \mathrm{Hom}_E(N^i, \prod_{i=t-j} \underline{\mathrm{Hom}}_k(E(-j), M^t_{-j})) \\
(**) &\cong \prod_{i\in\mathbb{Z}} \mathrm{Hom}_E(N^i, \bigoplus_{i=t-j} \underline{\mathrm{Hom}}_k(E(-j), M^t_{-j})) \\
&\cong \prod_{i\in\mathbb{Z}} \mathrm{Hom}_E(N^i, \bigoplus_{i=t+j} \underline{\mathrm{Hom}}_k(E(j), M^t_{j})) \\
&\cong \prod_{i\in\mathbb{Z}} \mathrm{Hom}_E(N^i, \mathbf{R}(M)^i) \supset \mathrm{Hom}_{Com(E)}(\mathbf{N}, \mathbf{R}(\mathbf{M}))
\end{aligned}
$$

Let $f \in \mathrm{Hom}_{Com(S)}(\mathbf{L}(\mathbf{N}), \mathbf{M})$, so that $f := \{\{f^i_j\}_{i+j=t} : \bigoplus_{i+j=t} S(j) \otimes_k N^i_j \to M^t\}_{t\in\mathbb{Z}}$. We track f through $\mathcal{L}$ to then check $\mathcal{L}(f)$ respects the differential in the category of complexes of E-modules as we desire. The unmarked isomorphisms above are basically identities, i.e. identities on the domain or codomain, and don't affect f so we omit them.

(1) We start by applying the following isomorphism. Let R be a ring, let B be a left R-module, and let $(A_i)_{i\in I}$ be an indexed family of left R-modules, then there is an isomorphism,

$$
\mu : \mathrm{Hom}_R(\bigoplus_{i\in I} A_i, B) \to \prod_{i\in I} \mathrm{Hom}_R(A_i, B),
$$

$$
\mu : h \mapsto \{h\alpha_i : A_i \to B\}_{i\in I} \text{ such that } \alpha_i \text{ is the inclusion of } A_i \to \bigoplus_{i\in I} A_i.
$$

With the inverse, $\mu^{-1} : \{h_i\}_{i\in I} \mapsto \{h_i p_i\}_{i\in I} := h$ where p_i is the projection of $\bigoplus_{i\in I} A_i \to A_i$

When we apply this isomorphism to our case we have the following,

$$\mu : \prod_{t\in\mathbb{Z}} \mathrm{Hom}_S(\bigoplus_{i+j=t} S(j)\otimes_k N^i_{-j}, M^t) \to \prod_{t\in\mathbb{Z}}\prod_{i+j=t} \mathrm{Hom}_S(S(j)\otimes_k N^i_{-j}, M^t).$$

(2) Now we apply the Hom-tensor adjunction:

$$T : \prod_{t\in\mathbb{Z}}\prod_{i+j=t} \mathrm{Hom}_S(S(j)\otimes_k N^i_{-j}, M^t) \to \prod_{t\in\mathbb{Z}}\prod_{i+j=t} \mathrm{Hom}_k(N^i_{-j}, \underline{\mathrm{Hom}}_S(S(j), M^t)).$$

(3) Now we apply the isomorphism $\underline{\mathrm{Hom}}_s(S(j), M^t) \cong M^t(-j)$ in the second component. We have a similar isomorphism for the case of modules but now we need to deal with the degree. Define $\phi : \underline{\mathrm{Hom}}_s(S(j), M^t) \to M^t(-j)$ where $\phi(p) = p(1_{S(j)})$. Since p is a degree respecting map and $1 \in S(j)$ is in degree $-j$ implies $p(1)$ is in degree $-j$. Meaning we need to shift elements in M^t up by degree j, and hence $p(1) \in M^t(-j)$. Similarly we define $\phi^{-1} : m \mapsto (q_m(s) = sm)$ for fixed $m \in M^n(j)$, $\forall s \in S(j)$. Now to see that they are actually inverses of each other $\phi^{-1}\phi(p(s)) = \phi^{-1}(p(1)) = s(p(1)) = p(s)$ $\forall s \in S(-j)$ and $\phi\phi^{-1}(m) = \phi(p_m(s)) = p_m(1) = (1)m = m$. We now define our isomorphism:

$$\Phi : \prod_{t\in\mathbb{Z}}\prod_{i+j=t} \mathrm{Hom}_k(N^i_{-j}, \underline{\mathrm{Hom}}_S(S(j), M^t)) \to \prod_{t\in\mathbb{Z}}\prod_{i+j=t} \mathrm{Hom}_k(N^i_{-j}, M^t(-j)).$$

(4) Here we apply the isomorphism γ to the first component of Hom, $\gamma : N^i \otimes_E E(-j) \to N^i(-j)$ where $\gamma : n \otimes e \mapsto ne$. Here n is not twisted and e has degree twisted by j. Similarly,$\gamma^{-1} : N^i(-j) \to N^i \otimes_E E(-j)$ where $\gamma^{-1} : n \mapsto n \otimes 1_{E(-j)}$. The degree is preserved by gamma since $1_{E(-j)}$ has degree j. Now to see that they are inverses, $\gamma^{-1}(\gamma(n\otimes e)) = \gamma^{-1}(ne) = ne \otimes 1 = (n \otimes 1)e = n \otimes e$ and $\gamma(\gamma^{-1}(n)) = \gamma(n \otimes 1) = (n)(1) = n$. We define the following isomorphism:

$$\Gamma : \prod_{t\in\mathbb{Z}}\prod_{i+j=t} \mathrm{Hom}_k(N^i(-j), M^t_j) \to \prod_{t\in\mathbb{Z}}\prod_{i+j=t} \mathrm{Hom}_k(N^i \otimes_E E(-j), M^t_j).$$

(5) Now we apply Hom-tensor adjunction one last time.

$$T^{-1}: \prod_{t\in\mathbb{Z}} \prod_{i+j=t} \mathrm{Hom}_k(N^i \otimes_E E(-j), M_j^t) \to \prod_{n\in\mathbb{Z}} \prod_{i+j=t} \mathrm{Hom}_s(N^i, \underline{\mathrm{Hom}}_k(E(-j), M_j^t)).$$

(*) Here we reindex the products.

(6) Following a similar set up as (1) we use the following isomorphism,

$$\nu : \prod_{i\in I} \mathrm{Hom}(A, B_i) \cong \mathrm{Hom}(A, \prod_{i\in I} B_i).$$

$$\nu : \{h_i\}_{i\in I} \mapsto \{\beta_i h_i\}_{i\in I} := h, \text{ such that } \beta_i \text{ is the inclusion of } B_i \to \prod_{i\in I} B_i.$$

When we apply this isomorphims to our case we have the following,

$$\nu : \prod_{i\in\mathbb{Z}} \prod_{t-j=i} \mathrm{Hom}_E(N^i, \underline{\mathrm{Hom}}_k(E(-j), M_j^t)) \to \prod_{i\in\mathbb{Z}} \mathrm{Hom}_E(N^i, \prod_{i=t-j} \underline{\mathrm{Hom}}_k(E(-j), M_j^t)).$$

(**) Given graded R-modules $\{M_i\}_{i\in I}$, their product is the graded module $P = \bigoplus_{m\in\mathbb{Z}} P_m$ where $P_m = \prod_{i\in I}(M_i)_m$. Here is the case we care about. Fix $i \in \mathbb{Z}$, $\prod_{t-j=i} \mathbf{R}(M^t)^j = \bigoplus_{t-j=i} \mathbf{R}(M^t)^j$. Below we prove it:

Proof. It suffices to show: $\prod_{n-j=i} \mathbf{R}(M^t)^j_m = \bigoplus_{n-j=i} \mathbf{R}(M^t)^j_m \forall m \in \mathbb{Z}$. That is, given only $m \in \mathbb{Z}$, there exists only finitely many pairs (t, j) such that $t - j = i$ and $\mathbf{R}(M^t)^j_m \neq 0$.

Let's check this: $\mathbf{R}(M^t)^j_m = \underline{\mathrm{Hom}}_k(E(j), M_j^t)_m$. Then note that M_j^t is in internal degree 0 here. Thus, $\underline{\mathrm{Hom}}_k(E(j), M_j^t)$ lives in degrees $-j, -j-1, ..., -j-(t+1)$. So given $m \in \mathbb{Z}$, $\underline{\mathrm{Hom}}_k(E(j), M_j^t)_m \neq 0$ if and only if $j \in \{-m, ..., -m-(t+1)\}$. Thus, the only pairs (t, j) such that $t - j = i$ and $\mathbf{R}(M^t)^j_m \neq 0$ are $(i-m, -m), ..., (i-m-(t+1), -m-(t+1))$. $\square$

So we have that $\mathcal{L}\{f_j^i\}_{i+j=t}(n)(e) = \{f_j^i(1 \otimes ne)))\}_{t-j=i}$. Now we confirm that the map respects the differential, i.e. it commutes, $\mathcal{L}(\{f_j^i\}_{i+j=t+1})(\partial_N^t(n)) - \partial_{\mathbf{R}}^t(\mathcal{L}(\{f_j^i(n)\}_{i+j=t})) = 0$.

$$\mathcal{L}(\{f_j^i\}_{i+j=t+1})(\partial_N^t(n)) - \partial_{\mathbf{R}}^t(\mathcal{L}(\{f_j^i(n)\}_{i+j=t})) =$$

$$= \{f_j^i(1 \otimes \partial_N^i(n)e)\}_{i+1=t-j} - \{(-1)^i \phi_j^i(f_j^i(1 \otimes ne) + \partial_M^i(f_j^i(1 \otimes ne)\}_{i+1=t-j}$$

$$
\begin{aligned}
&= \{f_j^i(1 \otimes \partial_N^i(n)e)\}_{i+1=t-j} - \{(-1)^i \Sigma_l^k x_l(f_j^i(e_l(1 \otimes ne))\}_{i+1=t-j} - \{\partial_M^i(f_j^i(1 \otimes ne)\}_{i+1=t-j} \\
&= \{f_j^i(1 \otimes \partial_N^i(n)e)\}_{i+1=t-j} - \{(-1)^i \Sigma_l^k x_l(f_j^i(e_l(1 \otimes ne))\}_{i+1=t-j} - \{f_j^i(\partial_{\mathbf{L}}^i(1 \otimes ne))\}_{i+1=t-j} \\
&= \{f_j^i(1 \otimes \partial_N^i(n)e)\}_{i+1=t-j} - \{(-1)^i \Sigma_l^k x_l(f_j^i(e_l(1 \otimes ne))\}_{i+1=t-j} \\
&\qquad - \{(f_j^i((-1)^{i+1}\psi(1 \otimes ne)) + (1 \otimes \partial_N^i(n)e))\}_{i+1=t-j} \\
&= \{f_j^i(1 \otimes \partial_N^i(n)e)\}_{i+1=t-j} - \{(-1)^i \Sigma_l^k x_l(f_j^i(e_l(1 \otimes ne))\}_{i+1=t-j} \\
&\qquad - \{(-1)^{i+1} f_j^i(\Sigma_l^k(x_l(1 \otimes e_l ne))\}_{i+1=t-j} - \{f_j^i(1 \otimes \partial_N^i(n)e))\}_{i+1=t-j} \\
&= -\{(-1)^i \Sigma_l^k x_l(f_j^i(e_l(1 \otimes ne))\}_{i+1=t-j} + \{(-1)^i \Sigma_l^k x_l(f_j^i(e_l(1 \otimes ne))\}_{i+1=t-j} \\
&= 0
\end{aligned}
$$

Hence we have shown that given a chain map f, $\mathcal{L}(f)$ is a chain map, but notice that that the above argument can be traced backwards and shows that given a chain map $\mathcal{L}(f)$, then f is a chain map.

Lastly notice that naturality is clear as the following diagram commutes as required for $f : N_1 \to N_2$ in Com(E) and $g : M_1 \to M_2$ in Com(S),

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Com}(S)}(\mathbf{L}(\mathbf{N}_2), \mathbf{M}_1) \arrow[r, "\cong"] \arrow[d, "\mathrm{Hom}_{\mathrm{Com}(S)}(\mathbf{L}(f),g)"'] & \mathrm{Hom}_{\mathrm{Com}(E)}(\mathbf{N}_2, \mathbf{R}(\mathbf{M}_1)) \arrow[d, "\mathrm{Hom}_{\mathrm{Com}(E)}(f,\mathbf{R}(g))"] \\
\mathrm{Hom}_{\mathrm{Com}(S)}(\mathbf{L}(\mathbf{N}_1), \mathbf{M}_2) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathrm{Com}(E)}(\mathbf{N}_1, \mathbf{R}(\mathbf{M}_2)).
\end{tikzcd}
$$

□

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Appendix A

Example

Let k be a field. Consider a vector space W generated by three elements, x_0, x_1, x_2. Let V be the dual of W, generated by e_0, e_1, e_2. Again let $E = \bigwedge V$ and $S = \text{Sym}(W)$. Consider E as an E-module over itself. Similarly, let S be as an S-module. Recall that these are naturally graded algebras and modules. Then the complex $\mathbb{L}(E)$ and $\mathbb{R}(S)$ have the form,

$$\mathbb{L}(E) : 0 \to S(0) \otimes_k E_0 \to S(1) \otimes_k E_{-1} \to S(2) \otimes_k E_{-2} \to S(3) \otimes_k E_{-3} \to 0,$$

$$\mathbb{R}(S) : 0 \to \underline{\text{Hom}}_k(E(0), S_0) \to \underline{\text{Hom}}_k(E(1), S_1) \to \underline{\text{Hom}}_k(E(2), S_2) \to \underline{\text{Hom}}_k(E(3), S_3) \to \dots.$$

Notice that $S(0) \cong k$ and $E(0) \cong k$. In general for a vector space generated by n variables the lengths of the complex of $\mathbb{L}$ will be $n+1$ since $e_i \otimes e_i = 0$. So, the only element of degree $n+1$ is $e_0 \otimes ... \otimes e_n$ and anything that has greater degree would have a square.

Appendix B

Macaulay2 Example

The bgg function in Macaulay2 is only the right adjoint $\mathbb{R}$. It takes a finitely generated graded S-module M and returns the ith map in the complex $\mathbb{R}(M)$. Here is the same example from above that has been copied over from Macaulay2.

```
i1 : loadPackage "BGG";

i2 : S = ZZ/101[x_0..x_2];

i3 : E = ZZ/101[e_0..e_2, SkewCommutative => true];

i4 : M = S^1;

i5 : bgg(0,M,E)

o5 = {-1} | e_0 |

     {-1} | e_1 |

     {-1} | e_2 |

               3        1

o5 : Matrix E  <--- E

i6 : bgg(1,M,E)

o6 = {-2} | e_0 0   0   |

     {-2} | e_1 e_0 0   |

     {-2} | e_2 0   e_0 |

     {-2} | 0   e_1 0   |

     {-2} | 0   e_2 e_1 |

     {-2} | 0   0   e_2 |
```

```
             6       3
o6 : Matrix E  <--- E
i7 : bgg(2,M,E)
o7 = {-3} | e_0 0   0   0   0   0   |
     {-3} | e_1 e_0 0   0   0   0   |
     {-3} | e_2 0   e_0 0   0   0   |
     {-3} | 0   e_1 0   e_0 0   0   |
     {-3} | 0   e_2 e_1 0   e_0 0   |
     {-3} | 0   0   e_2 0   0   e_0 |
     {-3} | 0   0   0   e_1 0   0   |
     {-3} | 0   0   0   e_2 e_1 0   |
     {-3} | 0   0   0   0   e_2 e_1 |
     {-3} | 0   0   0   0   0   e_2 |
             10       6
o7 : Matrix E   <--- E
i8 : bgg(3,M,E)
o8 = {-4} | e_0 0   0   0   0   0   0   0   0   0   |
     {-4} | e_1 e_0 0   0   0   0   0   0   0   0   |
     {-4} | e_2 0   e_0 0   0   0   0   0   0   0   |
     {-4} | 0   e_1 0   e_0 0   0   0   0   0   0   |
     {-4} | 0   e_2 e_1 0   e_0 0   0   0   0   0   |
     {-4} | 0   0   e_2 0   0   e_0 0   0   0   0   |
     {-4} | 0   0   0   e_1 0   0   e_0 0   0   0   |
     {-4} | 0   0   0   e_2 e_1 0   0   e_0 0   0   |
     {-4} | 0   0   0   0   e_2 e_1 0   0   e_0 0   |
     {-4} | 0   0   0   0   0   e_2 0   0   0   e_0 |
     {-4} | 0   0   0   0   0   0   e_1 0   0   0   |
     {-4} | 0   0   0   0   0   0   e_2 e_1 0   0   |
     {-4} | 0   0   0   0   0   0   0   e_2 e_1 0   |
```

```
      {-4} | 0    0    0    0    0    0    0    0    e_2 e_1 |

      {-4} | 0    0    0    0    0    0    0    0    0    e_2 |

               15         10

o8 : Matrix E   <--- E
```