3-Cycle Systems and Structure within Graph Decompositions

by

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Abstract

An H-decomposition of a graph G is a partition of the edge set $E(G)$ such that each element of the partition induces a subgraph isomorphic to H. A packing or cover of λK_n (with triples) is an ordered pair (V, B) where V is an n-element set and B is a set of 3-element subsets of V called blocks such that each 2-element subset of V appears in at most λ blocks or at least λ blocks respectively. Define $E(B) = \{\{x, y\}, \{x, z\}, \{y, z\} \mid \{x, y, z\} \in B\}$. The leave of a packing is defined to be the multiset of edges $L = E(\lambda K_n) - E(B)$ and the excess or padding of a cover is defined to be the multiset of edges $P = E(B) - E(\lambda K_n)$.

In this dissertation, necessary and sufficient conditions for the existence of K_3-decompositions of $K = \lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$ are found when $\lambda_1 \geq \lambda_2$, vastly generalizing the results in the literature on K_3-decompositions of K. In a specific case of this problem (namely when $n = 2$), it is useful to know for which simple quadratic subgraphs Q of K_n (so Q cannot have 2-cycles) do there exist a K_3-decomposition of $\lambda K_n - E(Q)$ (that is, a packing of λK_n with leave equal to $E(Q)$). A complete solution to this question is provided; in addition to being useful in proving the first result, it is also significant in that it extends a classic result of Colbourn and Rosa who answered the same question when $\lambda = 1$.

In terms of the quadratic leave problem, the previous result, while short and simple, has a gap in that it does not allow Q to have 2-cycles; the next result resolves this issue. In a packing of $2K_n$, the neighborhood graph of a vertex v is defined to be the graph induced by the multiset of edges $\{\{a, b\} \mid \{a, b, v\} \in B\}$. In a maximum packing of $2K_n$, the neighborhood graph of a vertex is a 2-regular graph on either $n - 1$ or $n - 2$ vertices. Colbourn and Rosa provided a chararterization of which 2-regular graphs on $n - 1$ or $n - 2$ vertices can be the neighborhood graph of a vertex in some maximum packing of $2K_n$ when $n \equiv 0$ or 1 (mod 3); this dissertation provides such a characterization in the case where

$n \equiv 2 \pmod 3$. This result along with the Colbourn and Rosa result ($n \equiv 0$ or $1 \pmod 3$) is used to find necessary and sufficient conditions for a K_3-decomposition of $\lambda K_n - E(Q)$ where Q is any 2-regular graph on at most n vertices (so Q can have 2-cycles).

Finally, having already found necessary and sufficient conditions for Q to be a 2-regular leave of λK_n, the problem of when a quadratic graph Q has edge set equal to the excess of a cover of λK_n is considered, and necessary and sufficient conditions for a K_3-decomposition of $\lambda K_n + E(Q)$ appear in the dissertation.

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Chapter 1

Introduction

This dissertation will focus on two main problems. First, it will look at finding K_3-decompositions of a specific family of graphs. Second, it will look at structure within packings and covers of λK_n. This introduction will first give a number of very general definitions in Section 1.1, and then discuss some previously solved problems related to the results in this dissertation in Section 1.2. The remaining chapters of this dissertation will be devoted to proving the results I have obtained over the past five years.

1.1 Definitions

A graph G is an ordered pair $(V(G), E(G))$ where $V(G)$ is a set of elements called vertices and $E(G)$ is a multiset of unordered pair of vertices; the elements of $E(G)$ are called edges. When $|V(G)| = n$, for notational purposes, it is often convenient to let $V(G) = \mathbb{Z}_n = \{0, 1, \ldots, n-1\}$. A graph H is said to be a subgraph of a graph G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. Two particular graphs that appear extensively in this dissertation are λK_n, which is the graph with n vertices in which every pair of vertices is joined by λ edges, and $\lambda_1 K_n \vee_{\lambda_2} \lambda_1 K_m$, which is the graph in which the vertex set is partitioned into two parts M and N where $|M| = m$, $|N| = n$, and each pair of vertices is joined by λ_1 edges if they are in the same part and is joined by λ_2 edges if they are in different parts. Two vertices u and v are said to be adjacent if $\{u, v\} \in E(G)$. If u and v are adjacent, we write $u \sim v$. The edge $\{u, v\}$ is said to be incident with the vertices u and v. The degree of a vertex v, denoted by $deg_G(v) = deg(v)$, is the number of edges incident with v. A k-cycle is a graph on k vertices in which every vertex has degree 2. When a cycle C is considered as a subgraph of K_n, C is said to be a Hamilton cycle if $|V(C)| = |V(K_n)| = n$ and C is said to be a near-Hamilton cycle if $|V(C)| = n - 1$. Figures 1.1,1.2, and 1.3 show the graphs $2K_9$, $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$, and C_8 respectively.

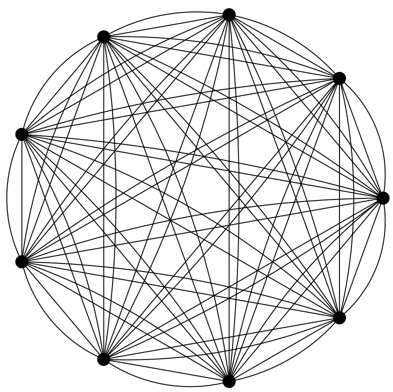

Figure 1.1: $2K_9$

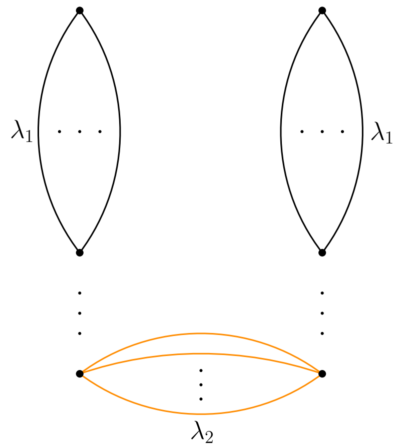

Figure 1.2: $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$

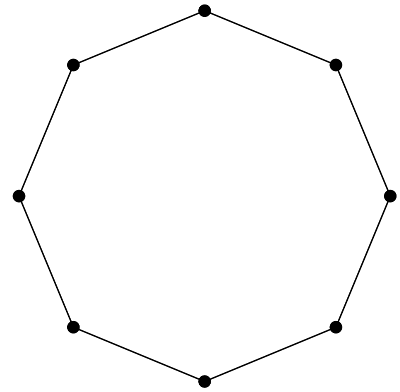

Figure 1.3: C_8

1.2 History

An H-decomposition of a graph G is a partition of the edge set $E(G)$ such that each element of the partition induces a subgraph isomorphic to H. In this dissertation, K_3-decompositions of graphs will be studied. Perhaps the most famous problem related to this topic is finding necessary and sufficient conditions for the existence of a K_3-decomposition of K_n. This problem was solved by Kirkman who showed in [19] that there exists a K_3-decomposition of K_n if and only if $n \equiv 1$ or $3 \pmod 6$. A natural extension of this problem is to find necessary and sufficient conditions for the existence of a K_3-decomposition of λK_n. This was solved by Hanani in [17] where it was shown that there exists a K_3-decomposition of λK_n if and only if $\lambda(n-1)$ is even, $\lambda(n)(n-1)$ is divisible by 3, and $n \neq 2$. A λ-fold triple system is an ordered pair (V, B) where V is a n-element set and B is a set of 3-element subsets of V called blocks such that each 2-element subset of V appears in λ blocks of B. The blocks of B are also called triples. It is straightforward to see that a λ-fold triple system is equivalent to a K_3-decomposition of λK_n.

With the necessary and sufficient conditions for a K_3-decomposition of λK_n well-established, there are a few natural directions to go, three of which are as follows:

1. What are necessary and sufficient conditions for K_3-decompositions of more general graphs G?
2. What type of structure is present in K_3-decompositions of λK_n?
3. What are some natural notions of closeness to K_3-decompositions of λK_n, and when can they be achieved?

These three questions are the main focus of the work in this dissertation. The first question is studied almost exclusively in Chapter 2; hence terminology specific to that question will appear in Chapter 2. The other questions are studied throughout the dissertation, and hence the rest of this introduction will provide some definitions and history related to the last two questions.

The two most natural notions of closeness to a K_3-decomposition of a graph G are packings and coverings. For the purposes of this dissertation, a packing of a graph G is a K_3-decomposition of a subgraph H of G. In the case where $G = \lambda K_n$, a packing is sometimes referred to as a partial (λ-fold) triple system. If (V, B) is a (partial) triple system, define the set of edges $E(B) = \{\{x, y\}, \{x, z\}, \{y, z\} \mid \{x, y, z\} \in B\}$. If (V, B) is a packing of λK_n, then the leave of the packing is defined to be the multiset of edges $L = E(\lambda K_n) \setminus E(B)$. It will cause no confusion to also refer to the leave as being the subgraph induced by $E(\lambda K_n) \setminus E(B)$. In particular, in the special case where $L = \{\{a, b\}, \{a, b\}\}$, L is expressed as the 2-cycle (a, b). A vertex v is said to be in the leave if there is some edge in the leave that is incident with v. A maximum packing is a packing such that among all packings the number of edges in its leave is as small as possible (with respect to the graph G). On the other hand, a cover of a graph G is a K_3-decomposition of $E(G) + P$ where P is a multiset of edges with underlying vertex set $V(G)$; in this dissertation, the topic of covers will only appear when $G = \lambda K_n$. If (V, B) is a cover of λK_n, define the set of edges $E(B) = \{\{x, y\}, \{x, z\}, \{y, z\} \mid \{x, y, z\} \in B\}$; the excess or padding of the cover is defined to be $E(B) \setminus E(\lambda K_n)$. Much like with leaves, it is commonplace to look at the subgraph induced by the excess as opposed to the excess itself. A minimum cover is a cover such that among all covers the number of edges in its excess is as small as possible (with respect to the graph G).

Maximum packings and minimum covers have also been well studied, and the leaves of all maximum packings of λK_n and the excesses of all minimum covers of λK_n have been found; for instance, see [23]. In this dissertation, quadratic leaves and excesses will be studied, but before getting to this problem, a related problem is considered.

In any partial triple system (V, B) of $2K_n$, the neighborhood of a vertex $v \in V$ is the graph induced by $\{\{x, y\} \mid \{v, x, y\} \in B\}$. If (V, B) is a maximum packing of $2K_n$ then the neighborhood of each vertex is a 2-regular graph, also called a quadratic graph. It will cause no confusion to also refer to the neighborhood as a set of cycles, each being a component of the neighborhood. It is natural to ask for which quadratic graphs Q does there exist a maximum packing (V, B) of $2K_n$ in which there exists a vertex $v \in V$ for which the neighborhood of v is Q? Colbourn and Rosa came up with a large part of the answer in [9].

Theorem 1.1. [9] *Suppose $n \equiv 0$ or $1 \pmod 3$. A 2-regular graph Q on $n-1$ vertices is the neighborhood of a vertex in a 2-fold triple system on n vertices if and only if $(n, Q) \notin \{(6, C_2 \cup C_3), (7, C_3 \cup C_3)\}$.*

One purpose of looking at this structure within 2-fold triple systems is that it helps determine when 2-fold triple systems are not isomorphic. The dissertation extends this result by proving an analogous result for $n \equiv 2 \pmod 3$; this result appears in Chapter 3. A simpler proof of this result that also subsumes the Colbourn and Rosa result appears in Chapter 4.

This structure problem seems unrelated to the earlier topic of quadratic leaves and excesses, but in fact, the topic of quadratic leaves is related to this structure problem. In most cases, deleting a vertex in a maximum packing of $2K_n$ induces a packing of $2K_{n-1}$ where the leave induces a quadratic graph. For quadratic excesses, there is perhaps not as clear a motivation for studying them, but they do form a natural complement to quadratic leaves and are hence discussed. In terms of quadratic leaves and excesses, the only explicit results prior to this dissertation are for $\lambda = 1$ and are as follows:

Theorem 1.2. [11] *Let Q be a 2-regular (quadratic) simple graph. There exists a K_3-decomposition of $K_n - E(Q)$ if and only if*

1. *n is odd,*

2. *$|E(K_n)| - |E(Q)|$ is divisible by 3, and*

3. *$(n, Q) \notin \{(7, C_3 \cup C_3), (9, C_4 \cup C_5)\}$.*

Theorem 1.3. *[10] Let Q be a quadratic graph on n vertices. Then Q is the excess of a cover of K_n if and only if*

1. *n is odd, and*

2. *$|E(Q)| + |E(K_n)| \equiv 0 \pmod 3$.*

In this dissertation, both of these results are extended to the case where $\lambda > 1$, although it should be mentioned that part of the the quadratic leave result for when $\lambda = 2$ is a direct consequence of Theorem 1.1. The result for quadratic leaves appears in Chapter 3 although a partial result is mentioned in Chapter 2, while the result for quadratic excesses appears in Chapter 5.

More specific terminology will be introduced in the chapters as it is needed and the dissertation now goes into the specific results obtained over the last five years.

Chapter 2

Group Divisible Designs with Two Associate Classes and Quadratic Leaves of Triple Systems

2.1 Introduction

A group divisible design with two associate classes $GDD(v, \{g_1, ..., g_n\}, k, \lambda_1, \lambda_2)$ is an ordered triple (V, G, B) where V is a v-set of symbols, G is a partition of V into n sets (called groups) of sizes $g_1, ..., g_n$ (possibly $g_i = g_j$ for some $i \neq j$ so $\{g_1, ..., g_n\}$ is regarded as a multiset), and B is a set of k-element subsets of V known as blocks, such that any two distinct elements of V appear together in λ_1 blocks if they are in the same group and λ_2 blocks otherwise. In the special case where $\lambda_1 = 0$, $\lambda_2 = 1$, and $k = 3$, it is common to use $GDD(v, \{g_1, ..., g_n\})$ instead of $GDD(v, \{g_1, ..., g_n\}, 3, 0, 1)$, and the design is called a group divisible design.

Bose and Shimamoto were among the first to classify such designs [2]. Fu, Rodger, and Sarvate solved the existence of these designs for $k = 3$ when all groups have the same size [15, 16]. A more general problem is to settle the existence question for $k = 3$ when the groups have different sizes. In general this is a difficult problem, perhaps the most difficult case being when there are exactly two groups. Pabhapote and Punnim solved the existence in this case under the assumptions that $\lambda_2 = 1$ and that neither m nor n is 2 [25]. In this chapter, their result is generalized with a different proof, completely solving the interesting case where one of the two groups has size 2 (see Section 2.5) and also solving the case where $\lambda_2 \leq \lambda_1$ (see Section 2.3). These results are stated in the major theorem of the chapter (see Theorem 2.2).

The problem can be approached in terms of an equivalent graph decomposition problem. Let $K = K(M, N, \lambda_1, \lambda_2) = \lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$ where $|M| = m$ and $|N| = n$ be a graph on the vertex set $M \cup N$ (M and N are called the parts) in which for each $x, y \in M \cup N$ with $x \neq y$, there exist λ_1 edges between x and y if x and y are in the same part and λ_2 edges between x and y otherwise. An edge $e = \{x, y\}$ is said to be pure if x and y lie in the same part and is said to be mixed otherwise. A K_3-decomposition of a graph G is an ordered pair $(V(G), B)$ where B is a partition of $E(G)$ into sets, each of which induces a K_3; in this chapter focus will be placed on the case where $G = K$. Sometimes the copies of K_3 are called triples and the K_3-decomposition of K is called a triple system of K. It is straightforward to see that such a graph decomposition is equivalent to a $GDD(v = m + n, \{m, n\}, 3, \lambda_1, \lambda_2)$. To begin, some obvious necessary conditions for the existence of this K_3-decomposition of K are given. Some of these conditions appear in other papers (such as [13]), but the proof is included here for completeness.

Lemma 2.1. *Let $K = \lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$ with $|M| = m$ and $|N| = n$. Assume that $n = 2$ only if $m = 2$. The following conditions are necessary for the existence of a K_3-decomposition of K.*

1. *3 divides $\lambda_1(\binom{m}{2} + \binom{n}{2}) + \lambda_2 mn$,*
2. *2 divides $\lambda_1(m-1) + \lambda_2 n$ and 2 divides $\lambda_1(n-1) + \lambda_2 m$,*
3. *$2\lambda_1(\binom{m}{2} + \binom{n}{2}) \geq \lambda_2 mn$,*
4. *if $m = 2$ then $\lambda_1 \leq \lambda_2 n$, and*
5. *if $m = 2$ and $n \leq 2$ then $\lambda_1 = \lambda_2$.*

Proof. The first and second conditions are necessary since the total number of edges must be divisible by 3 and the degree of each vertex must be divisible by 2 respectively. The last three conditions come from looking at the types of triples that can be chosen. Note that every triple that contains a mixed edge must use precisely two mixed edges and a pure edge; thus, the third condition is necessary. If $m = 2$ then each triple that uses a pure edge in M, must also use two mixed edges; thus, the fourth condition is necessary. Finally, if $m = 2$ and n is less than or equal to 2 then every triple must use a pure edge and two mixed edges, so the fifth condition is necessary. $\square$

Note that under the assumption that $\lambda_1 \geq \lambda_2$, the third necessary condition simplifies to requiring that either $m > 1$ or $n > 1$.

It seems reasonable to conjecture that the conditions of Lemma 2.1 are sufficient as well. As noted earlier, this conjecture is proved in the case where $m = 2$ (see Section 2.5: Theorem 2.14) and where $\lambda_1 \geq \lambda_2$ (see Section 2.3: Theorem 2.11). These proofs culminate in the following result.

Theorem 2.2. *Let $K = \lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$ with $m = 2$ if $n = 2$. Let $\lambda_1 \geq \lambda_2$ if $m \neq 2$. There exists a K_3-decomposition of K if and only if*

1. *3 divides $\lambda_1(\binom{m}{2} + \binom{n}{2}) + \lambda_2 mn$,*
2. *2 divides $\lambda_1(m-1) + \lambda_2 n$ and 2 divides $\lambda_1(n-1) + \lambda_2 m$,*
3. *$2\lambda_1(\binom{m}{2} + \binom{n}{2}) \geq \lambda_2 mn$,*
4. *if $m = 2$ then $\lambda_1 \leq \lambda_2 n$, and*
5. *if $m = 2$ and $n \leq 2$ then $\lambda_1 = \lambda_2$.*

Another valuable result in this chapter is the generalization of the classic result by Colbourn and Rosa establishing necessary and sufficient conditions for the existence of a K_3-decomposition of $K_n - E(Q)$ where Q is a 2-regular subgraph of K_n (see Theorem 2.12). In Section 2.4, the generalization of this problem to $\lambda K_n - E(Q)$ is completely solved as long as Q is simple, the necessary and sufficient conditions depending on n, Q, and λ (see Theorem 2.13). This result is then used in the proof of Lemma 2.18, thus helping to settle the $m = 2$ case (see Theorem 2.14).

2.2 Terminology

The following terminology will be used throughout the chapter so it is defined here and complements the terminology given in Chapter 1. In K_v with vertex set $\mathbb{Z}_v$, define the difference of an edge joining two vertices i and j as $d(i,j) = \min\{|i-j|, v-|(i-j)|\}$. For each $D \subset \{1, ..., v-1\}$, let $G_v(D)$ be the graph with vertex set Z_v and edge set consisting of all edges of differences in D. A difference d is said to be *good* if $\frac{v}{gcd(v,d)}$ is even. A difference triple is a triple (a,b,c) of unique integers from the set $\{1, ..., v-1\}$, such that $a+b=c$ or $a+b+c=v$.

For $i \in \{1,3,5\}$, let $F_i(V(G))$ be a graph with vertex set $V(G)$ in which one vertex has degree i and the rest have degree 1; these graphs are known as 1-factors, 3-poles (tripoles), and 5-poles when $i = 1, 3$, and 5 respectively.

A graph G is said to be evenly equitable if each vertex of G has even degree and for all $u, v \in V(G)$, $|d(u) - d(v)| \leq 2$.

2.3 The Case where K has no parts of size 2

In this section, Theorem 2.2 is proved in the case where neither part has size 2 (see Theorem 2.11). First some lemmas that are used in the proof of the theorem are given. The first addresses the well-known existence of maximum packings and minimum covers of μK_v with triples.

Theorem 2.3. *[17, 23] Let $\mu \geq 1$ and $v \geq 3$. Let P (or L) be any multigraph with the least number of edges in which all vertices have degree congruent to $\mu(v-1)$ (mod 2) and with $|E(P)| + \frac{\mu v(v-1)}{2} \equiv 0$ (mod 3) (or $\frac{\mu v(v-1)}{2} - |E(L)| \equiv 0$ (mod 3) respectively). Then there exists a K_3-decomposition of $\mu K_v \cup E(P)$ (or $\mu K_v - E(L)$ respectively).*

A latin square L of order n is an $n \times n$ array containing the symbols $0, 1, \ldots, n-1$ such that each symbol appears exactly once in each row and each column. If L is a latin square, we refer to the symbol in cell (a, b) as being $a \circ b$. With the operation $\circ$, the latin square is referred to as a quasigroup; the quasigroup is denoted by $(L, \circ)$. A quasigroup is said to be idempotent if $i \circ i = i$ for every $i \in \{0, 1, \ldots, n-1\}$. The next lemma shows that quasigroups can be constructed so that they are not commutative; this result will be used later to build a triple system with a desired property.

Lemma 2.4. *For all $n \geq 4$, there exists an idempotent quasigroup of order n that is not commutative. That is, for all $n \geq 4$, there exists an idempotent quasigroup of order n in which there exist a and b such that $a \circ b \neq b \circ a$.*

Proof. If n is even, then the result follows since in every idempotent quasigroup, each symbol appears once on the main diagonal and hence appears off the main diagonal an odd number of times. If n is odd, then define $i \circ i = i$ and $(i+1) \circ (j+2) = i \circ j$ for $0 \leq i, j < n$ (reducing sums modulo n). Since 2 is relatively prime to n, this product defines a quasigroup. Since $n \geq 4$, cells $(1, 0)$ and $(0, 1)$ contain different symbols. □

The following useful observation is easily proved.

Observation 2.5. *For any $m, n, \lambda \geq 1$, if the number of edges in both λK_n and λK_m are not congruent (mod 3), then the number of edges in one of the parts is congruent to 0 (mod 3).*

Proof. If neither $|E(\lambda K_m)|$ nor $|E(\lambda K_n)|$ is divisible by 3, then $|E(K_m)| \equiv |E(K_n)| \equiv 1$ (mod 3) and then $|E(\lambda K_m)| \equiv |E(\lambda K_n)| \equiv \lambda$ (mod 3) which proves the claim. □

The next lemma is also easily proved and helps to eliminate cases in the proof of Theorem 2.11.

Lemma 2.6. *Let λ_1, λ_2, m, and n be positive integers. Furthermore, suppose that 3 divides $(\lambda_1(\binom{m}{2} + \binom{n}{2}) + \lambda_2 mn)$. If $m + n \equiv 2$ (mod 3) and $\lambda_2 \equiv 1$ or 2 (mod 3), then either m or n is congruent to 0 (mod 3).*

Proof. Assume that $m \equiv n \equiv 1 \pmod 3$. Then $\binom{m}{2} \equiv \binom{n}{2} \equiv 0 \pmod 3$. But none of λ_2, m, n is congruent to 0 (mod 3). So $\lambda_2 mn$ is not congruent to 0 (mod 3) and thus $(\lambda_1(\binom{m}{2} + \binom{n}{2}) + \lambda_2 mn)$ is not divisible by 3, a contradiction. □

Lemma 2.7. *Let $v \equiv 0$ or 1 (mod 3), λ be even, and $v \geq 12$. Then there exists a K_3-decomposition $(V, T(z_0, z_1, z_2, z_3, z_4))$ of λK_v containing the following triples: $\{z_0, z_1, z_2\}$, $\{z_0, z_1, z_3\}$, and $\{z_2, z_3, z_4\}$.*

Proof. Since $v \equiv 0$ or $1 \pmod 3$ and $\lambda - 2$ is even, by Theorem 2.3, let (V, B_0) be a K_3-decomposition of $(\lambda-2)K_v$. Let $\epsilon \in \{0,1\}$ with $\epsilon \equiv v \pmod 3$. Let $V = (\mathbb{Z}_{\frac{v-\epsilon}{3}} \times \mathbb{Z}_3) \cup I$ where $I = \{\infty\}$ if $\epsilon = 1$ and $I = \emptyset$ otherwise. Let $(L, \circ)$ be a non-commutative latin square of order $\frac{v-\epsilon}{3} \geq 4$ (see Lemma 2.4). For each $x \in \mathbb{Z}_{\frac{v-\epsilon}{3}}$ let $(\{x\} \times \mathbb{Z}_3, B_x)$ or $(\{\infty\} \cup (\{x\} \times \mathbb{Z}_3), B_x)$ be a K_3-decomposition of $2K_{3+\epsilon}$ if $\epsilon = 0$ or 1 respectively. Let $B' = \{\{(i,l), (j,l), (i \circ j, l+1)\}, \{(i,l), (j,l), (j \circ i, l+1)\} \mid 0 \leq i < j \leq \frac{v-\epsilon}{3}, 0 \leq l \leq 2\}$ where $l+1 = 0$ when $l = 2$. Let $B = \cup_{x \in \mathbb{Z}_{\frac{v-\epsilon}{3}}} B_x \cup B'$. Then (V, B) is a K_3-decomposition of $2K_v$. Furthermore, since $a \circ b \neq b \circ a$ for some $a, b \in \mathbb{Z}_{\frac{v-\epsilon}{3}}$, define $(z_0, ..., z_4) = ((a,0), (b,0), (a \circ b, 1), (b \circ a, 1), ((a \circ b) \circ (b \circ a), 2))$. The result is then seen in $(V, B_0 \cup B)$. □

To settle two small cases in Lemma 2.10 (see below), the two following well-known results are needed, with one on 1-factorizations of difference-induced subgraphs, and the other on the existence of GDDs.

Theorem 2.8. *[20] Let D be a set of differences. Then $G_v(D)$ has a 1-factorization if and only if D contains at least one good difference.*

The second theorem is proved in much greater generality in the original paper, but for the purposes of this chapter, it states the following:

Theorem 2.9. *[8] There exists a $GDD(6k, \{12, 6, 6, ..., 6\})$ for all $k \geq 5$.*

The following lemma is useful in a couple of cases in the proof of the main theorem of this chapter.

Lemma 2.10. *Let $\lambda_2 = 1$. Let $v \equiv 0$ or $4 \pmod 6$ with $v \geq 12$ and let $z(v) \in \{3, 4\}$ with $z(v) \equiv v \pmod 3$. Then there exists a K_3-decomposition of $\lambda_2 K_v - E(L)$ where L is a graph on v vertices with $z(v)$ vertices of degree 3 and the rest of degree 1 such that no two vertices of degree 3 are adjacent.*

Proof. The cases $v = 12$ and $v = 16$ are handled first and are also used in the next paragraph. For $v = 12$, the result is seen in $(\mathbb{Z}_4 \times \mathbb{Z}_3, B_{12} = \{\{(i, 0), (j, 1), (i + j, 2)\} \mid 0 \leq i, j \leq 3\} \cup \{\{(1, j), (2, j), (3, j)\} \mid j \in \mathbb{Z}_3\})$ where $i + j$ is reduced (mod 4). For $v = 16$, consider $(\mathbb{Z}_{16}, B' = \{\{i, i + 2, i + 5\}, \{i, i + 6, i + 7\} \mid 0 \leq i \leq 15\})$ where addition is done (mod 16). Note that these triples contain all edges of differences $1, 2, 3, 5, 6,$ and 7. Now note that the differences of 4 and 8 form four disjoint copies of K_4. Let B'' consist of one triple from each K_4. Then $(\mathbb{Z}_{16}, B_{16} = B' \cup B'')$ is the required K_3-decomposition. For each $v \in \{12, 16\}$, the leave L'_v consists of $z(v)$ vertex-disjoint stars.

Now assume $v \geq 30$ and let $\epsilon \in \{0, 4\}$ with $\epsilon \equiv v \pmod 6$. Let $\delta = \frac{v-\epsilon-12}{6}$; hence, $\delta \geq 3$. By Theorem 2.9, let (V, B) be a GDD with δ groups of size 6 on the vertex sets $G_1, ..., G_\delta$ and one group of size 12 with vertex set G_0. Let $I = \{\infty_1, \infty_2, \infty_3, \infty_4\}$ if $v \equiv 4 \pmod 6$ and $I = \emptyset$ otherwise. If $v \equiv 4 \pmod 6$, then for $1 \leq i \leq \delta$, let $B_i = \{\{\infty_j, x, y\} \mid 1 \leq j \leq 4, \{x, y\} \in H_{j,i}, \{H_{1,i}, ..., H_{5,i}\}$ is a 1-factorization of K_6 on the vertex set $G_i\}$ and let $L_i = H_{5,i}$. If $v \equiv 0 \pmod 6$, then for $1 \leq i \leq \delta$, let B_i consist of the triples of a maximum packing on the vertex set G_i with leave the 1-factor L_i. For each $v \in \{12, 16\}$, let $(G_0 \cup I, B_0)$ be a K_3-decomposition of $K_v - L_0$ where L_0 is isomorphic to L'_v (as defined in the previous paragraph). Then $(\cup_{i \in \mathbb{Z}_{\delta+1}} G_i \cup I, \cup_{i \in \mathbb{Z}_{\delta+1}} B_i)$ is the required K_3-decomposition with leave $\cup_{i \in \mathbb{Z}_{\delta+1}} L_i$.

It remains to settle the result for $v \in \{18, 22, 24, 28\}$.

Let $v \in \{22, 28\}$, and let $I = \{\infty_1, ..., \infty_{v-16}\}$. By Theorem 2.8, there exists a 1-factorization $\{F_1, ..., F_{v-16}\}$ of $G_{16}(\{2, 3, 5\})$ and $G_{16}(\{1, 2, 3, 5, 6, 7\})$ if $v = 22$ or 28 respectively (since 3 is a a good difference). Let $B' = \{\{i, i + 6, i + 7\} \mid 0 \leq i \leq 15\}$ (where addition is done modulo 16) if $v = 22$ and $B' = \emptyset$ if $v = 28$. Let B'' be the set of triples of a maximum packing on the vertex set I (with leave a 1-factor in both cases (see Theorem 2.3)). Let $B = B' \cup B'' \cup \{\{4+i, 8+i, 12+i\} \mid i \in \mathbb{Z}_4\} \cup \{\{x, y, \infty_j\} \mid \{x, y\} \in F_j, 1 \leq j \leq v-16\}$. Then $(\mathbb{Z}_{16} \cup I, B)$ is the required packing.

Let $v = 18$. Consider the vertex set $\{\infty_1, ..., \infty_6\} \cup \mathbb{Z}_{12}$. By Theorem 2.8, let $\{F_1, ..., F_6\}$ be a 1-factorization of $G_{12}(\{1, 2, 5\})$. Let $(\{\infty_1, ..., \infty_6\}, B_1)$ be a maximum packing of K_6 with leave the 1-factor L. Let $B = B_1 \cup \{\{x, y, \infty_j\} \mid \{x, y\} \in F_j, 1 \leq j \leq 6\} \cup \{\{i, i+4, i+8\} \mid 0 \leq i \leq 3\} \cup \{\{i + 3, i + 6, i + 9\} \mid 0 \leq i \leq 2\}$ where addition is done modulo 12. Then $(\{\infty_1, ..., \infty_6\} \cup \mathbb{Z}_{12}, B)$ is the required decomposition with leave consisting of L together with 3 vertex disjoint stars joining vertex i to vertices $i + 3, i + 6$, and $i + 9$ for each $i \in \mathbb{Z}_3$.

Let $v = 24$. Consider the vertex set $\{\infty_1, ..., \infty_{10}\} \cup \mathbb{Z}_{14}$. By Theorem 2.8, let $\{F_1, ..., F_9\}$ be a 1-factorization of $G_{14}(\{3, 4, 5, 6, 7\})$ (with good difference 3). Now, let $F_{10} = \{\{13, 11\}, \{12, 0\}, \{1, 3\}, \{2, 4\}, \{5, 7\}, \{6, 8\}, \{9, 10\}\}$. This matching along with the set of triples $\{\{3i, 3i + 1, 3i + 2\} \mid 0 \leq i \leq 1\} \cup \{\{3i + 1, 3i + 2, 3i + 3\} \mid 2 \leq i \leq 3\}$, the star joining 13 to $0, 1$, and 12, and the star joining 6 to $4, 5$, and 7 use up all edges of difference 1 and 2 except $\{2, 3\}, \{8, 10\}$, and $\{9, 11\}$. Let $(\{\infty_1, ..., \infty_{10}\}, B_1)$ be a maximum packing of K_{10} with leave L a tripole. Let $B = B_1 \cup \{\{x, y, \infty_j\} \mid \{x, y\} \in F_j, 1 \leq j \leq 10\} \cup \{\{3i, 3i + 1, 3i + 2\} \mid 0 \leq i \leq 1\} \cup \{\{3i + 1, 3i + 2, 3i + 3\} \mid 2 \leq i \leq 3\}$. Then $(\{\infty_1, .., \infty_{10}\} \cup \mathbb{Z}_{14}, B)$ is the required decomposition with leave consisting of L, the star joining 13 to $0, 1$, and 12, the star joining 6 to $4, 5$, and 7, and the independent edges $\{2, 3\}, \{8, 10\}$, and $\{9, 11\}$. $\square$

The next theorem is the main result of this section.

Theorem 2.11. *Let λ_1 and λ_2 be positive integers with $\lambda_1 \geq \lambda_2$. Let m and n be positive integers not equal to 2. Then there exists a K_3-decomposition of $\lambda_1 K_n \vee_{\lambda_2} \lambda_1 K_m$ if and only if*

1. *3 divides $\lambda_1(\binom{m}{2} + \binom{n}{2}) + \lambda_2 mn$,*
2. *2 divides $\lambda_1(m-1) + \lambda_2 n$ and 2 divides $\lambda_1(n-1) + \lambda_2 m$, and*
3. *either $m \neq 1$ or $n \neq 1$.*

Proof. The necessity of Conditions $(1-3)$ was shown in Lemma 2.1. To prove the sufficiency, regular use of Theorem 2.3 will be made with $\mu = \lambda_2$ and $v = m + n$; this is possible since $\lambda_2 \geq 1$ by assumption and since $m + n \geq 3$ by Condition 3. Several cases and subcases will be considered in turn.

Case 1: Suppose that $\lambda_2(m+n-1)$ is even (so each vertex of $\lambda_2 K_{m+n}$ has even degree). This will be broken into two subcases.

Case 1.1: Suppose the number of edges of $\lambda_2 K_{m+n}$ is divisible by 3.

By Theorem 2.3, let $(M \cup N, T_1)$ be a K_3-decomposition of $\lambda_2 K_{m+n}$. Now applying Observation 2.5 with $\lambda = \lambda_1 - \lambda_2$ along with Condition (1) shows that the number of edges in each of $(\lambda_1 - \lambda_2)K_m$ and $(\lambda_1 - \lambda_2)K_n$ is divisible by 3. The second condition shows that each vertex of $(\lambda_1 - \lambda_2)K_m$ and of $(\lambda_1 - \lambda_2)K_n$ has even degree. Hence, by Theorem 2.3 there exist K_3-decompositions (M, T_2) and (N, T_3) of $(\lambda_1 - \lambda_2)K_m$ and of $(\lambda_1 - \lambda_2)K_n$ respectively. Then $(M \cup N, \cup_{i=1}^{3} T_i)$ is the required K_3-decomposition. Note Case 1.1 settles the existence for Case 1 unless both $\lambda_2 \not\equiv 0 \pmod 3$ and $m + n \equiv 2 \pmod 3$.

Case 1.2: Suppose $\lambda_2 \equiv 1$ or $2 \pmod 3$ and that $(m + n) \equiv 2 \pmod 3$.
By Lemma 2.6, one of m and n is divisible by 3, say $m \equiv 0 \pmod 3$. So by Condition 1, $\lambda_1 \equiv 0 \pmod 3$. By Theorem 2.3, let $(M \cup N, T_1)$ be a maximum packing of $\lambda_2 K_{m+n}$ with leave L being a double edge joining two vertices n_1 and n_2 in N if $\lambda_2 \equiv 1 \pmod 3$ and being the four cycle (n_1, n_2, n_3, n_4) in N if $\lambda_2 \equiv 2 \pmod 3$. (In this case, since $n + m \equiv 2 \bmod 3$,

since $m \equiv 0 \bmod 3$, and since $n \neq 2$, $n \geq 5$.) By Condition 2, each vertex of $(\lambda_1 - \lambda_2)K_m$ and of $(\lambda_1 - \lambda_2)K_n + E(L)$ has even degree. By Theorem 2.3, let (M, T_2) be a K_3-decomposition of $(\lambda_1 - \lambda_2)K_m$. Finally, by Theorem 2.3, let (N, T_3) be a minimum cover of $(\lambda_1 - \lambda_2)K_n$ with padding $P = L$, (recall that $\lambda_1 \equiv 0 \pmod 3$). Then $(M \cup N, \cup_{i=1}^{3} T_i)$ is the required K_3-decomposition.

Case 2: Now suppose that $\lambda_2(m + n - 1)$ is odd. Note that Condition 2 implies that m, n, and λ_1 are all even. This is broken into four subcases.

Case 2.1: Suppose that:

1. $m, n \equiv 0 \pmod 6$,
2. $m \equiv 0 \pmod 6$ and $n \equiv 4 \pmod 6$, or
3. $m, n \equiv 4 \pmod 6$.

By Theorem 2.3, there exists a maximum packing, $(M \cup N, T_1)$, of $\lambda_2 K_{m+n}$ with leave $L = F_1(M) \cup F_1(N)$, $F_1(M) \cup F_3(N)$, $F_3(M) \cup F_3(N)$ in cases (1), (2), and (3) respectively (note that both $|M|$ and $|N|$ are even so that each edge of the leave can lie entirely within M or N). By Theorem 2.3 let (M, T_2) be a minimum cover of $(\lambda_1 - \lambda_2)K_m$ with padding $L[M]$ (the subgraph of L induced by M). By Theorem 2.3, let (N, T_3) be a minimum cover of $(\lambda_1 - \lambda_2)K_n$ with padding $L[N]$. Then $(M \cup N, \cup_{i=1}^{3} T_i)$ is the required K_3-decomposition.

Case 2.2: Let $m \equiv 0 \pmod 6$ and $n \equiv 2 \pmod 6$.

Note that the necessary conditions imply that $\lambda_1 \equiv 0 \pmod 6$ (Condition 1 implies $\lambda_1 \equiv 0 \pmod 3$ and Condition 2 implies λ_1 is even.) By Theorem 2.3, let $(M \cup N, T_1)$ be a maximum packing of $\lambda_2 K_{m+n}$ with leave $F(M) \cup F(N)$, $F(M) \cup F_3(N)$, or $F(M) \cup F_5(N)$ for $\lambda_2 \equiv 1, 3$, or $5 \pmod 6$ respectively. By Theorem 2.3, let (M, T_2) be a minimum cover of $(\lambda_1 - \lambda_2)K_m$ with padding $F(M)$. Note that $(\lambda_1 - \lambda_2) \equiv 5, 3$, or $1 \pmod 6$ respectively. By Theorem 2.3, let (N, T_3) be a minimum cover of $(\lambda_1 - \lambda_2)K_n$ with padding $F(N), F_3(N)$, or $F_5(N)$ respectively. Then $(M \cup N, \cup_{i=1}^{3} T_i)$ is the required K_3-decomposition.

For the next two subcases, the structure from Lemmas 2.7 and 2.10 are useful here.

Case 2.3: Let $m, n \equiv 2 \pmod 6$.

In Case 2.3, two cases are considered at first and then merged at the end.

First suppose $\lambda_2 = 1$. Since $2 \notin \{m, n\}$, $m, n \geq 8$, so since in this subcase $m + n \equiv 4 \pmod 6$, by Lemma 2.10 there exists a K_3-decomposition $(M \cup N, T_1')$ of $K_{m+n} - (E(L_M) \cup E(L_N))$ where L_M and L_N are spanning subgraphs of M and N respectively, each consisting of two vertex disjoint copies of $K_{1,3}$ and a matching on the remaining vertices.

Since λ_2 is odd in Case 2, we can now suppose that $\lambda_2 \geq 3$. By Lemma 2.7, there exists a K_3-decomposition $(M \cup N, T(m_1, m_2, n_1, n_2, n_3))$ of $(\lambda_2 - 1)K_{m+n}$, where $\{m_1, m_2\} \subset M$ and $\{n_1, n_2, n_3\} \subset N$. By Theorem 2.3, there exists a maximum packing $(M \cup N, T_1^*)$ of K_{m+n} with leave $F(M) \cup F_3(N)$; name them so they contain the edges $\{m_1, m_2\}$, $\{n_1, n_2\}$, $\{n_3, n_6\}$, $\{n_3, n_5\}$, and $\{n_3, n_4\}$. Let $T_1 = T_1' \cup T_1^* \cup \{\{n_1, n_2, m_1\}, \{n_1, n_2, m_2\}\} \backslash \{\{n_1, n_2, n_3\}, \{m_1, m_2, n_1\}, \{m_1, m_2, n_2\}\}$. Then $(M \cup N, T_1)$ is a packing of $\lambda_2 K_{m+n}$ with leave $L_M \cup L_N$ where L_M contains three copies of the edge $\{m_1, m_2\}$ and a matching on the remaining vertices and L_N is a five-pole with n_3 being the vertex of degree 5.

In either case, triples are added to $(M \cup N, T_1)$ as follows: Let (M, T_2) and (N, T_3) be minimum covers of M and N with padding L_M and L_N respectively (see Theorem 2.3). Then $(M \cup N, \cup_{i=1}^{3} T_i)$ is the required K_3-decomposition.

Case 2.4: Let $m \equiv 4 \pmod 6$ and $n \equiv 2 \pmod 6$.

First suppose $\lambda_2 = 1$. Since in this subcase $m + n \equiv 0 \pmod 6$ and since $n \neq 2$ so $n \geq 8$ and $m + n \geq 12$, by Lemma 2.10 there exists a K_3-decomposition $(M \cup N, T_1')$ of $K_{m+n} - (E(L_M) \cup E(L_N))$ where L_M is a spanning subgraph of M consisting of a $K_{1,3}$ and a matching on the remaining vertices and L_N is a spanning subgraph of N consisting of two vertex disjoint copies of $K_{1,3}$ (recall that we observed that $n \geq 8$) and a matching on the remaining vertices.

Again since λ_2 is odd in Case 2, we can now suppose that $\lambda_2 \geq 3$. By Lemma 2.7, there exists a K_3-decomposition $(M \cup N, T(m_1, m_2, n_1, n_2, n_3))$ of $(\lambda_2 - 1)K_{m+n}$, where $\{m_1, m_2\} \subset$

M and $\{n_1, n_2, n_3\} \subset N$. By Theorem 2.3, there exists a maximum packing $(M \cup N, T_1^*)$ of K_{m+n} with leave $F(M) \cup F(N)$; name them so they contain the edges $\{m_1, m_2\}$ and $\{n_1, n_2\}$. Let $T_1 = T_1' \cup T_1^* \cup \{\{n_1, n_2, m_1\}, \{n_1, n_2, m_2\}\} \backslash \{\{n_1, n_2, n_3\}, \{m_1, m_2, n_1\}, \{m_1, m_2, n_2\}\}$. Then $(M \cup N, T_1)$ is a packing of $\lambda_2 K_{m+n}$ with leave $L_M \cup L_N$ where L_M contains three copies of the edge $\{m_1, m_2\}$ and a matching on the remaining vertices and L_N is a tripole.

In either case, triples are added to $(M \cup N, T_1)$ as follows. Let (M, T_2) and (N, T_3) be minimum covers of M and N with padding L_M and L_N respectively (see Theorem 2.3). Then $(M \cup N, \cup_{i=1}^3 T_i)$ is the required K_3-decomposition.

□

2.4 Quadratic Leaves

In this section, the following strong result of Rosa and Colbourn is generalized to all values of λ. Not only is this of interest in its own right, but it will also be useful in Section 2.5.

A graph is said to be *quadratic* if each vertex has either degree 2 or degree 0.

Theorem 2.12. *[11]*

1. *Let Q be a quadratic (simple) graph on $n \equiv 3 \pmod 6$ vertices and $c \equiv 0 \pmod 3$ edges. Then Q is the leave of a packing of K_n unless $n = 9$ and $Q = C_4 \cup C_5$.*

2. *Let $n \equiv 1$ or $5 \pmod 6$, and let Q be a quadratic graph on $c < n$ edges with $c \equiv n - 1 \pmod 3$. Then Q is the leave of a packing of K_n unless $n = 7$ and $Q = C_3 \cup C_3$.*

The next result is the main result of the section. A portion of the result for $\lambda = 2$ was proved in [9].

Theorem 2.13. *Let $n > 2$ and let Q be a quadratic simple graph on at most n vertices. Then Q is the leave of a packing of λK_n if and only if:*

1. *Either λ is even or n is odd,*

2. 3 *divides* $|E(\lambda K_n)| - |E(Q)|$, *and*

3. *(a) if* $\lambda = 1$ *and* $n = 7$, *then* $Q \neq C_3 \cup C_3$,

(b) if $\lambda = 1$ *and* $n = 9$, *then* $Q \neq C_4 \cup C_5$, *and*

(c) if $\lambda = 2$ *and* $n = 6$, *then* $Q \neq C_3 \cup C_3$.

Proof. Suppose there exists a packing of $\lambda K_n - E(Q)$. Since each vertex of Q has even degree and each vertex of $\lambda K_n - E(Q)$ must have even degree, each vertex of λK_n must have even degree so Condition 1 is necessary. Since the edges of $\lambda K_n - E(Q)$ are partitioned into sets of size 3, Condition 2 is necessary.

Conditions 3(a) and 3(b) follow from Theorem 2.12. Condition 3(c) follows from the fact that the only K_3-decomposition of $2K_6$ (up to isomorphism) does not contain two disjoint triples. Thus Condition 3 is necessary.

The proof of the sufficiency is broken into several cases, based on the congruence classes (mod 6) of n. Let n, λ, and Q satisfy Conditions $(1 - 3)$. Theorem 2.12 handles the case where $\lambda = 1$, so assume $\lambda > 1$, Also, if $|E(Q)| = 0$, then by Theorem 2.3, there exists a K_3-decomposition of λK_n, which proves the result. Hence, assume $|E(Q)| \geq 3$. (Note that no simple quadratic graph has 1 or 2 edges.)

If $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$, Condition 2 is satisfied if and only if $|Q| \equiv 0 \pmod 3$. So, unless $\lambda > 1$ and $(n, Q) \in \{(7, C_3 \cup C_3), (9, C_4 \cup C_5)\}$, the result follows by taking a triple system of $K_n - E(Q)$ (see Theorem 2.12) together with a triple system of $(\lambda - 1)K_n$.

If $\lambda > 1$ and $(n, Q) = (7, C_3 \cup C_3)$, then by Theorem 2.3, the vertices of two triple systems of K_7 with vertex set V can be named to contain the triples $T_1 = \{a_1, b_1, c_1\}$ and $T_2 = \{a_2, b_2, c_2\}$ respectively, with $T_1 \cap T_2 = \emptyset$; let B_1 be the union of the triples of these triple systems with $Q = T_1 \cup T_2$ removed. If $\lambda > 1$ and $(n, Q) = (9, C_4 \cup C_5)$, then by Theorem 2.12, let (V, B_1') be a triple system of $K_9 - L_1$ where L_1 is the nine-cycle $(0, 1, 2, 3, 4, 5, 6, 7, 8)$ and let (V, B_1'') be a triple system of $K_9 - L_2$ where L_2 is the six-cycle $(0, 3, 6, 4, 8, 7)$. Let

$B_1 = B_1' \cup B_1'' \cup \{\{3,4,6\},\{0,7,8\}\}$. In either case, let (V, B_2) be a triple system of $(\lambda-2)K_n$. Then $(V, B_1 \cup B_2)$ is the required decomposition.

Suppose $n \equiv 5 \pmod 6$. Let $\epsilon \in \{1,2,3\}$ with $\epsilon \equiv \lambda \pmod 3$. By Theorem 2.3, let (V, B_1) be a K_3-decomposition of $(\lambda - \epsilon)K_n$. The three ϵ values are now considered separately.

Suppose $\epsilon = 1$. By Theorem 2.12, let (V, B_2) be a K_3-decomposition of $K_n - E(Q)$.

Suppose $\epsilon = 2$. Let Q' be formed from Q by replacing one cycle, $c = (0, 1, ..., x)$, of length $x+1 \geq 4$ in Q with the cycle $(1, ..., x)$ (by Condition 2, $|E(Q)| \equiv |E(2K_n)| \equiv 2 \pmod 3$ and hence there must be a cycle of length at least 4). By Theorem 2.12, let (V, B_2') be a K_3-decomposition of $K_n - E(Q')$. Let (V, B_2'') be a K_3-decomposition of $K_n - E(L)$ where L is the 4-cycle $(0, 1, 2, x)$. Let $B_2 = B_2' \cup B_2'' \cup \{\{1, 2, x\}\}$.

Finally, suppose $\epsilon = 3$. By Condition 2, $|V(G[Q])| \leq n - 2$ so say $0 \notin V(G[Q])$. Let $c = (1, 2, ..., x)$ be any cycle in Q and set $c' = (0, 1, ..., x)$. Form Q' from Q by replacing c with c'. Also by Condition 2, $|E(Q)| \equiv |E(\lambda K_n)| \equiv 0 \pmod 3$, so $|E(Q')| \equiv 1 \equiv |E(K_n)| \pmod 3$. So by Theorem 2.12, let (V, B_2') be a K_3-decomposition of $K_n - E(Q')$. By Theorem 2.3, let (V, B_2'') be a K_3-decomposition of $2K_n - E(L)$ where L consists of two copies of the edge $\{1, x\}$. Let $B_2 = B_2' \cup B_2'' \cup \{\{0, 1, x\}\}$.

In each case, $(V, B_1 \cup B_2)$ is the required K_3-decomposition.

Suppose $n \equiv 4 \pmod 6$. First suppose $(n, Q) \neq (10, C_4 \cup C_5)$. By Condition 1, λ is even. By Theorem 2.3, let (V, B_1) be a K_3-decomposition of $(\lambda - 2)K_n$. By Condition 2, $|V(G[Q])| \leq n - 1$ so say $0 \notin V(G[Q])$. Let $G = G[V(K_n)\backslash\{0\}]$. By Theorem 2.12, let (V^*, B_2) be a K_3-decomposition of $G - E(Q)$. Let (V, B_3) be a maximum packing of K_n with leave L, a tripole, that includes the edges $\{0, 1\}, \{1, 2\}$, and $\{1, 3\}$. Finally, let $B_4 = \{\{x, y, 0\} \mid \{x, y\} \text{ is an independent edge in } L\} \cup \{\{0, 1, 2\}, \{0, 1, 3\}\}$. Then $(V, \cup_{i=1}^{4} B_i)$ is the required decomposition. Now suppose $(n, Q) = (10, C_4 \cup C_5)$. Let $(\mathbb{Z}_{10}, B_1)$ be a maximum packing of K_{10} with leave consisting of the edges $\{0, 1\}, \{0, 8\}, \{0, 9\}, \{2, 3\}, \{4, 5\}$, and $\{6, 7\}$, and let $(\mathbb{Z}_{10}, B_2)$ be a maximum packing with leave consisting of the edges $\{0, 3\}, \{1, 2\}$,

$\{5,6\},\{4,8\},\{7,8\},\{8,9\}$. Then $(V, B_1 \cup B_2 \cup \{\{0,8,9\}\})$ is the required packing of $2K_{10}$ and can be combined with a K_3-decomposition of $(\lambda-2)K_n$ to get the result for higher values of λ.

Suppose $n \equiv 2 \pmod 6$. Note that $n \neq 2$. By Condition 1, λ is even. Let $\epsilon \in \{2,4,6\}$ and set $\epsilon \equiv \lambda \pmod 6$. By Theorem 2.3, let (V, B_1) be a K_3-decomposition of $(\lambda-\epsilon)K_n$.

Suppose $\epsilon = 2$.

First suppose that Q contains a cycle of length at least 5. Let Q' be formed from Q by replacing a cycle, $c = (0,1,...,x)$, of length $x+1 \geq 5$ in Q with the cycle $c' = (2,...,x)$. Note that $0 \notin V(G[Q'])$. Let $G = G[V(K_n)\backslash\{0\}]$. Since $\lambda \equiv 2 \pmod 6$ (since $\epsilon = 2$ in this subcase), then $|E(Q)| \equiv |E(\lambda K_n)| \equiv 2 \pmod 3$, and hence $|E(Q')| \equiv 0 \pmod 3$. Further, $n-1 \equiv 1 \pmod 6$ so by Theorem 2.12, let (V^*, B_2) be a K_3-decomposition of $G - E(Q')$. Now let (V, B_3) be a maximum packing of K_n with leave L, a 1-factor, with $\{1,2\}$ and $\{0,x\}$ as edges in the 1-factor. Finally, let B_4 consist of the following triples: $\{\{x,y,0\} \mid \{x,y\} \in (L\backslash\{\{1,2\},\{0,x\}\})\} \cup \{\{0,2,x\}\}$. Then $(V, \cup_{i=1}^{4} B_i)$ is the required decomposition.

Now suppose Q contains only three and four cycles. There are at least two four cycles; name them $(0,1,2,3)$ and $(4,5,6,7)$. Let Q' be formed from Q by removing these two four cycles and replacing them with the six-cycle $(2,3,4,5,6,7)$. Note that $0 \notin V(G[Q'])$. Let $G = G[V(K_n)\backslash\{0\}]$. By the above argument, let (V^*, B_2) be a K_3-decomposition of $G - E(Q')$. Now let (V, B_3) be a maximum packing of K_n with leave L, a 1-factor, with $\{1,2\}$, $\{4,7\}$ and $\{0,3\}$ as edges in the 1-factor. Finally, let B_4 consist of the following triples: $\{\{x,y,0\} \mid \{x,y\} \in (L\backslash\{\{1,2\},\{0,3\},\{4,7\}\})\} \cup \{\{0,2,7\},\{0,3,4\}\}$. Then $(V, \cup_{i=1}^{4} B_i)$ is the required decomposition.

Suppose $\epsilon = 4$. By Condition 2, $|V(G[Q])| \leq n-1$ so say $0 \notin V(G[Q])$. Let $c = (1,2,...,x)$ be any cycle in Q and set $c' = (0,1,...,x)$. Form Q' from Q by replacing c with c'. By the above case for $\epsilon = 2$, let (V, B_2) be a packing of $2K_n$ with leave Q'. By Theorem

2.3, let (V, B_3) be a maximum packing of $2K_n$ with leave the double edge $\{1, x\}$. Then $(V, \cup_{i=1}^{3} B_i \cup \{\{0, 1, x\}\})$ is the required decomposition.

Suppose $\epsilon = 6$. By Condition 2, $|V(G[Q])| \leq n - 2$ so say $0 \notin V(G[Q])$. Let $c = (1, 2, ..., x)$ be any cycle in Q and set $c' = (0, 1, ..., x)$. Form Q' from Q by replacing c with c'. By the above case for $\epsilon = 4$, let (V, B_2) be a packing of $4K_n$ with leave Q'. By Theorem 2.3, let (V, B_3) be a maximum packing of $2K_n$ with leave the double edge $\{1, x\}$. Then $(V, \cup_{i=1}^{3} B_i \cup \{0, 1, x\})$ is the required decomposition.

Finally, suppose $n \equiv 0 \pmod 6$. The proof is similar to the case where $n \equiv 2$ (mod 6) with $\epsilon = 2$ unless Q consists entirely of 3-cycles. It is first assumed that Q contains a 5-cycle, then two edges are removed from a cycle of length at least 5, a vertex is deleted, and then the leave is modified just as in the case where $n \equiv 2 \pmod 6$ with $\epsilon = 2$. If there are no five cycles, but at least one four cycle, then there are necessarily three four cycles since $|E(Q)| \equiv 0 \pmod 3$. Take two of the four cycles and then proceed in the same manner as in the case where $n \equiv 2 \pmod 6$ with $\epsilon = 2$. Finally, if there are only 3-cycles, note that $n \geq 12$ (by assumption) and take a K_3-decomposition of λK_n that contains a parallel class ($\frac{n}{3}$ disjoint triples; take the B_x from Lemma 2.7 for example). $\square$

2.5 The case where $m = 2$

A proof of Theorem 2.2 in the case where $m = 2$ is given in this section. The proof relies on results which follow it, but is worth reading first before all the details potentially cloud the idea.

Theorem 2.14. *Let $m = 2$. Let λ_1, λ_2, and n be positive integers. Then there exists a K_3-decomposition of $\lambda_1 K_n \vee_{\lambda_2} \lambda_1 K_2$ if and only if*

1. *3 divides $\lambda_1(\binom{m}{2} + \binom{n}{2}) + \lambda_2 mn$,*

2. *2 divides $\lambda_1(m-1) + \lambda_2 n$ and 2 divides $\lambda_1(n-1) + \lambda_2 m$,*

3. *$2\lambda_1(\binom{m}{2} + \binom{n}{2}) \geq \lambda_2 mn$,*

4. $\lambda_1 \leq \lambda_2 n$*, and*

5. if $n \leq 2$ *then* $\lambda_1 = \lambda_2$*.*

Proof. The necessity of Conditions $(1-5)$ has been shown in Lemma 2.1, so the sufficiency is now proved. Let $M = \{m_0, m_1\}$.

For both $n = 1$ and $n = 2$, Condition 5 requires $\lambda_1 = \lambda_2$. If $n = 1$, then the sufficiency follows from taking λ_2 copies of a triple system on 3 vertices. If $n = 2$, then Condition 2 implies that λ_1 is even. Thus, the sufficiency follows from taking $\frac{\lambda_1}{2}$ copies of a 2-fold triple system of $2K_4$.

Suppose $n \geq 3$. The following three steps are taken, although they are dependent upon lemmas provided afterwards.

Step 1: Since $m = 2$, the number of mixed edges incident with each vertex of N is even. Together with Condition 2, this implies that each vertex in $\lambda_1 K_n$ has even degree. Let (N, B_1) be a K_3-decomposition of $\lambda_1 K_n - E(L)$ where L is a subgraph of K_n such that:

(a) $|E(L)| = \lambda_2 n - \lambda_1$,

(b) the subgraph induced by $E(L)$ is connected, and

(c) each vertex of L has even degree less than or equal to $2\lambda_2$.

The existence of L is shown in Lemma 2.18, but at least it is noted here that $\lambda_2 n - \lambda_1 \geq 0$ by Condition 3.

Step 2: Since the subgraph of K induced by $E(L)$, $G = K[E(L)]$, is connected (by (b)) and since all vertices in G have even degree (by (c)), there exists an Euler circuit E of G. Alternately color the edges of E with colors 0 and 1. By Condition 2, $\lambda_2 n - \lambda_1$ is even, so E has even length, so for each vertex w in G, the number of edges in G incident with w colored 0 equals the number colored 1. Let $B_2 = \{\{m_i, w, u\} \mid \{w, u\} \in E(L) \text{ is colored } i, i \in \mathbb{Z}_2\}$. Then for each $i \in \mathbb{Z}_2$ and each $n \in N$, the number of triples containing the pair $\{m_i, n\}$ is $\frac{d_G(n)}{2}$.

Step 3: By (c), $\lambda_2 - \frac{d_G(n)}{2}$ is a nonnegative integer for each $n \in N$. So let B_3 be the multiset of triples formed as follows: for each $n \in N$, let B_3 contain $\lambda_2 - \frac{d_G(n)}{2}$ copies of the triple $\{m_0, m_1, n\}$.

By (a), $\sum_{n \in N}(\lambda_2 - \frac{d_G(n)}{2}) = n\lambda_2 - |E(L)| = \lambda_1$, so λ_1 triples in B_3 contain the pair $\{m_0, m_1\}$. Therefore $(M \cup N, \cup_{i=1}^{3} B_i)$ is the required decomposition. □

So the result for $m = 2$ is proved provided that it can be shown that the K_3-decomposition of $\lambda_1 K_n - E(L)$ in Step 1 actually exists. Conditions 3 and 1 imply that the number of edges in $\lambda_1 K_n - E(L)$ is nonnegative and divisible by 3 respectively. It is now show how the K_3-decomposition from Step 1 can be achieved. However, before showing the existence of L, a powerful result due to Simpson is given along with a corollary that will frequently be used in showing the existence of L.

A Langford sequence of order n and defect d with $n > d$ is a sequence $L = (l_1, l_2, ..., l_{2n})$ of $2n$ integers satisfying the conditions

(a) for every $k \in \{d, d+1, ..., d+n-1\}$, there exist exactly two elements $l_i, l_j \in L$ such that $l_i = l_j = k$, and

(b) if $l_i = l_j = k$, then $|i - j| = k$.

A hooked Langford sequence of order n and defect d with $n > d$ is a sequence $L = (l_1, l_2, ..., l_{2n}, l_{2n+1})$ of $2n$ integers satisfying the conditions

(a) $l_{2n} = 0$

(b) for every $k \in \{d, d+1, ..., d+n-1\}$, there exist exactly two elements $l_i, l_j \in L \setminus \{l_{2n}\}$ such that $l_i = l_j = k$, and

(c) if $l_i = l_j = k$, then $|i - j| = k$.

Theorem 2.15. *[29]*

1. *A Langford sequence of order n and defect d exists if and only if*

(a) $n \geq 2d - 1$ *and*

(b) $n \equiv 0, 1$ *(mod 4) and* d *is odd, or* $n \equiv 0, 3$ *(mod 4) and* d *is even.*

2. A hooked Langford sequence of order n *and defect* d *exists if and only if*

(a) $n(n - 2d + 1) + 2 \geq 0$ *and*

(b) $n \equiv 2, 3$ *(mod 4) and* d *is odd, or* $n \equiv 1, 2$ *(mod 4) and* d *is even.*

Corollary 2.16. *Let* $n \geq 3$*,* $w \geq d + 3n$*,* $0 \leq \delta \leq n$*, and* $D_1 \subset \{d + i \mid i \in \mathbb{Z}_n\}$ *with* $|D_1| = n - \delta$*. If there exists a Langford sequence or a hooked Langford sequence of order* n *and defect* d*, then there exists* $D \subset \{d + i \mid i \in \mathbb{Z}_{3n}\}$ *with* $|D| = 3\delta$ *and* $D_1 \cap D = \emptyset$ *such that there exists a set of triples* B' *such that* $B = \{b + j \mid b \in B', j \in \mathbb{Z}_w\}$ *is a* K_3*-decomposition of* $G_w(D)$*.*

The triples in B' are said to generate the K_3-decomposition of $G_w(D)$.

Proof. Let $L = (l_1, ..., l_{2n})$ or $(l_1, ..., l_{2n-1}, 0, l_{2n+1})$. Let $B' = \{\{0, i+n, j+n\} \mid l_i = l_j, l_i \notin D'\}$ so that the triple $\{0, i + n, j + n\}$ contains edges of differences $n + i$, $j + i$ and $d(i, j)$. Then $(\mathbb{Z}_w, B)$ is the required decomposition. $\square$

The following useful result will also be used in the proof of Lemma 2.18.

Theorem 2.17. *[27] For* $n \neq 2$ *there exists an equitable partial triple system containing* v *triples for any* $1 \leq v \leq \mu(n, \lambda)$ *where*

$$\mu(n, \lambda) = \begin{cases} \lfloor \frac{n}{3} \lfloor \frac{\lambda(n-1)}{2} \rfloor \rfloor - 1 & \begin{array}{l} \textit{if } n \equiv 2 \pmod 6 \quad \textit{and } \lambda = 4 \pmod 6 \\ \textit{if } n \equiv 5 \pmod 6 \quad \textit{and } \lambda = 1 \textit{ or } 4 \pmod 6 \end{array} \\ \lfloor \frac{n}{3} \lfloor \frac{\lambda(n-1)}{2} \rfloor \rfloor & \textit{otherwise} \end{cases}$$

The following lemma shows the existence of the decomposition described in Step 1 of the proof of the main theorem of this section.

Lemma 2.18. *Let $m = 2$ and suppose the following five conditions hold:*

1. *3 divides $\lambda_1(\binom{m}{2} + \binom{n}{2}) + \lambda_2 mn$,*

2. *2 divides $\lambda_1(m-1) + \lambda_2 n$ and 2 divides $\lambda_1(n-1) + \lambda_2 m$,*

3. *$2\lambda_1(\binom{m}{2} + \binom{n}{2}) \geq \lambda_2 mn$,*

4. *$\lambda_1 \leq \lambda_2 n$, and*

5. *if $n \leq 2$ then $\lambda_1 = \lambda_2$.*

Then there exists a subgraph L of $\lambda_1 K_n$ with $\lambda_2 n - \lambda_1$ edges which is evenly equitable and connected for which there exists a K_3-decomposition of $\lambda_1 K_n - E(L)$.

Proof. This lemma is proved in several cases. The vertex set of K_n will be $\mathbb{Z}_n$. First small values of λ are considered before a general construction is provided. For the small cases, Corollary 2.16 is consistently used.

Case 1: First, suppose $\lambda_1 = 1$. Note that L being evenly equitable and having $\lambda_2 n - \lambda_1$ edges is equivalent to requiring that every vertex in the leave have degree $2\lambda_2$ except for a single vertex which has degree $2\lambda_2 - 2$; the exceptional vertex in the following constructions will be vertex $0, 1$ or 2. Conditions 1 and 2 imply that $n \equiv 1, 5 \pmod 6$ and $\lambda_2 \equiv 1 \pmod 6$. It can be assumed that $\lambda_2 > 1$ since otherwise, the problem simplifies to simply looking for a K_3-decomposition of K_{6x+3} or K_{6x+1} which are both known to exist. Set $\lambda_2 = 6z + 1$ with $z \geq 1$.

Suppose $n \equiv 5 \pmod 6$. It can be assumed that $n \geq 17$, since if $n = 5$ or 11, then Condition 3 in conjunction with the fact that $\lambda_2 \equiv 1 \pmod 6$ implies that $\lambda_2 = 1$. For $n = 6k + 5 \geq 17$, using edges of only differences 1 and 2, let $B_1 = \{\{3i, 3i+1, 3i+2\} \mid 0 \leq i \leq 2k+1\}$, where addition is done $\pmod{6k+5}$. Note that every symbol except 0 appears in exactly one triple; 0 appears in two. If $n = 17, 23, 29$, or 35, let $B_2' = \{\{0,3,8\},\{0,4,10\}\},\{\{0,3,9\},\{0,4,11\},\{0,5,13\}\},\{\{0,4,14\},\{0,7,16\},\{0,3,8\}\{0,6,17\}\},$

or $\{\{0,3,17\},\{0,5,11\},\{0,7,20\},\{0,9,19\},\{0,4,12\}\}$ respectively. Otherwise, by Corollary 2.16, let B_2' be a set of triples that generate a K_3-decomposition of $G_n(D)$ where $D = \{3,4,...,3k+2\}$.

First note that $k \geq 2z$, since by Condition 3, $2+n(n-1) \geq 2\lambda_2 n = (12z+2)n$, so since $n-1$ is an integer and $n > 2$ it follows that $n-1 \geq 12z+2$, so $6k+4 \geq 12z+2$ so $k \geq 2z$. Therefore, B_2 can be formed by removing $2z \geq 2$ triples from B_2', one of which contains the difference 4. So $|B_2| = |B_2'| - 2z = k - 2z \geq 0$. Then $(\mathbb{Z}_n, B_1 \cup \{b+j \mid j \in \mathbb{Z}_n, b \in B_2\})$ is the required K_3-decomposition, since if $v \in \mathbb{Z}_n \setminus \{0\}$, then $d_L(v) = (n-1) - 2 - 6(k-2z) = 12z+2 = 2\lambda_2$ and $d_L(0) = (n-1) - 4 - 6(k-2z) = 2\lambda_2 - 2$. So L is evenly equitable with $\lambda_2 n - \lambda_1$ edges. Since all edges of difference 4 are in L, L is connected.

Suppose $n \equiv 1 \pmod 6$. It can be assumed that $n \geq 19$, since if $n = 7$ or 13, then Condition 3 in conjunction with the fact that $\lambda_2 \equiv 1 \pmod 6$ implies that $\lambda_2 = 1$. For $n = 6k+1 \geq 19$, using edges of only differences 1, 2, and 3, form triples as follows. Let $\epsilon \in \{1,4,7\}$ with $\epsilon \equiv n \pmod 9$. Let $n = 9j+\epsilon$ and $B_1' = \{\{9i, 9i+1, 9i+3\}, \{9i+1, 9i+2, 9i+4\}, \{9i+2, 9i+3, 9i+5\}, \{9i+4, 9i+6, 9i+7\}, \{9i+5, 9i+7, 9i+8\}, \{9i+6, 9i+8, 9i+9\} \mid 0 \leq i \leq j-1\}$. Note that symbols 0 and $9j$ appear in one triple; symbols $1,...,9j-1$ appear in two.

If $\epsilon = 1$, then let $B_1'' = \{\{9j, 0, 2\}\}$. If $\epsilon = 4$, then let $B_1'' = \{\{9j, 9j+1, 9j+3\}, \{9j+1, 9j+2, 0\}, \{9j+2, 9j+3, 1\}\}$. If $\epsilon = 7$, then let $B_1'' = \{\{9j, 9j+1, 9j+3\}, \{9j+1, 9j+2, 9j+4\}, \{9j+2, 9j+3, 9j+5\}, \{9j+4, 9j+6, 0\}, \{9j+5, 9j+6, 1\}\}$.

In any case, set $B_1 = B_1' \cup B_1''$. Note that every symbol except y appears in exactly two triples; y appears in three triples, where $y = 2$ if $\epsilon = 1$ and $y = 1$ if $\epsilon = 4$ or 7.

If $n \geq 25$, then there exists a set B_2' of triples that generates a K_3-decomposition of $G_n(D = \{4,5,...,3k\})$. This follows from Corollary 2.16 if $n \geq 49$. If $n = 25, 31, 37$, or 43, let $B_2' = \{\{0,4,13\},\{0,7,15\},\{0,5,11\}\}$, $\{\{0,5,11\},\{0,4,17\},\{0,9,19\},\{0,7,15\}\}$, $\{\{0,4,17\}, \{0,7,18\},\{0,6,14\},\{0,5,15\},\{0,9,21\}\}$, or $\{\{0,11,23\},\{0,7,22\},\{0,4,13\},\{0,6,16\}, \{0,5,19\},\{0,8,25\}\}$ respectively.

Now use B_2' to define B_2. First note that $k-1 \geq 2z$, since by Condition 3, $2+n(n-1) \geq 2\lambda_2 n = (12z+2)n$, so since $n-1$ is an integer and $n > 2$ it follows that $n-1 \geq 12z+2$, so $6k \geq 12z+2$, so $k-1 \geq 2z$. Therefore if $n \geq 25$, then B_2 can be formed by removing $2z \geq 2$ triples from B_2', one of which contains the difference 4. So $|B_2| = |B_2'|-2z = (k-1)-2z \geq 0$. If $n = 19$, then $\lambda_2 \in \{1,7\}$ so it can be assumed that $\lambda_2 = 7$; this means that $k-1 = 2z$ so define $B_2 = \emptyset$ in this case. Then for $n \geq 19$, $(\mathbb{Z}_n, B_1 \cup \{b+j \mid j \in \mathbb{Z}_n, b \in B_2\})$ is the required K_3-decomposition, since if $v \in \mathbb{Z}_n \setminus \{y\}$, then $d_L(v) = (n-1)-4-6((k-1)-2z) = 12z+2 = 2\lambda_2$ and $d_L(y) = (n-1)-6-6((k-1)-2z) = 2\lambda_2-2$. So L is evenly equitable with $\lambda_2 n - \lambda_1$ edges. Since all edges of difference 4 are in L, L is connected.

Case 2: Now suppose $\lambda_1 = 2$. Then $n \equiv 1, 2, 4$, or $5 \pmod 6$. If $\lambda_2 = 2$, the result follows from Theorem 2.3, so assume $\lambda_2 > 2$. Note that L being evenly equitable and having $\lambda_2 n - \lambda_1$ edges is equivalent to requiring that every vertex in the leave have degree $2\lambda_2$ except for two vertices each of which has degree $2\lambda_2 - 2$.

Suppose $n \equiv 1$ or $5 \pmod 6$. Conditions 1 and 2 imply that $\lambda_2 \equiv 2 \pmod 6$. It can be assumed that $n \geq 11$ since if $n = 5$ or 7, then Condition 3 in conjunction with the fact that $\lambda_2 \equiv 2 \pmod 6$ implies that $\lambda_2 = 2$.

First suppose $n \geq 17$. By Case 1, there exists a K_3-decomposition $(\mathbb{Z}_n, B_1)$ of $K_n - E(L_1)$ where L_1 is a subgraph of K_n with $(\lambda_2-1)n-(\lambda_1-1)$ edges which is evenly equitable and connected. By Theorem 2.12, there exists a K_3-decomposition $(\mathbb{Z}_n, B_2)$ of $K_n - E(L_2)$ where L_2 is a subgraph of K_n consisting of a $n-1$ cycle such that the vertex not in the cycle has maximum degree in L_1. Then $(\mathbb{Z}_n, B_1 \cup B_2)$ is the required decomposition with $L = G[E(L_1) \cup E(L_2)]$.

Suppose $n = 11$ or 13. Condition 3 implies that $\lambda_2 = 2$ or 8, so it can be assumed that $\lambda_2 = 8$. For $n = 11$, let $B = \{\{3i+j, 3i+1+j, 3i+2+j\} \mid 0 \leq i \leq 3, 0 \leq j \leq 1\}$. Then $(\mathbb{Z}_n, B)$ is the required K_3-decomposition, since if $v \in \mathbb{Z}_n \setminus \{0,1\}$, then $d_L(v) = 2(10)-4 = 16 = 2\lambda_2$ and $d_L(0) = d_L(1) = 2(10)-6 = 2\lambda_2 - 2$. So L is evenly equitable with $\lambda_2 n - \lambda_1$ edges. Since all edges of difference 4 are in L, L is connected. For $n = 13$, let $B = \{\{0+j, 1+j, 3+$

$j\}, \{1+j, 2+j, 4+j\}, \{2+j, 3+j, 5+j\}, \{4+j, 6+j, 7+j\}, \{5+j, 7+j, 8+j\}, \{6+j, 8+j, 9+j\}, \{9+j, 10+j, 12+j\}, \{10+j, 11+j, 0+j\}, \{11+j, 12+j, 1+j\} \mid 0 \le j \le 1\}$. Then $(\mathbb{Z}_n, B)$ is the required K_3-decomposition, since if $v \in \mathbb{Z}_n \setminus \{1, 2\}$, then $d_L(v) = 2(12) - 8 = 16 = 2\lambda_2$ and $d_L(1) = d_L(2) = 2(12) - 10 = 2\lambda_2 - 2$. So L is evenly equitable with $\lambda_2 n - \lambda_1$ edges. Since all edges of difference 4 are in L, L is connected.

Now assume $n \equiv 2$ or $4 \pmod 6$. In this case, the half-difference is present and used only once. Instead of thinking of the differences as $1, 2, ..., \frac{n}{2}, 1, 2, ..., \frac{n}{2} - 1$, it is useful to think of them as $1, 2, ..., n-1$ since, for instance, the difference 1 can be thought of as a difference $n - 1$. In both cases, by Condition 1, $\lambda_2 \equiv 2 \pmod 3$, so let $\lambda_2 = 3z + 2$.

Suppose $n \equiv 2 \pmod 6$ with $n > 2$, and let $n = 6k + 2$. Although, it seems unnecessary at this point, $\lambda_2 = 2$ is allowed for cases where $n \equiv 2 \pmod 6$ (see Cases 4 and 5 for the use of this result). Using only differences $1, 2$, and $n - 1 = 1$, let $B_1 = \{\{i, i+1, i+2\} \mid 0 \le i \le n-1, i \equiv 0, 1 \pmod 3\}$. Then each vertex has degree 4 from the triples in B_1 except 0 and 1 which have degree 6. Now assume $n \ge 20$. Consider the differences in $D' = \{3, 4, ..., n-2\}$. Then $|D'| \equiv 1 \pmod 3$. Let $v = n - 2$ or $n - 3$ if $\frac{n-5}{3} \equiv 0, 1 \pmod 4$ or $2, 3 \pmod 4$ respectively. By Corollary 2.16, there exists a set of triples B_2' that generates a K_3-decomposition of $G_n(D = D' \setminus \{v\})$. Note that $2k - 1 \ge z$, since by Condition 3, $4 + 2n(n-1) \ge 2\lambda_2 n = (6z+4)n$, so since $n - 1$ is an integer and $n > 2$, $(n-1) \ge (3z+2)$, so $6k + 1 \ge 3z + 2$, so $2k - 1 \ge z$. Therefore, B_2 can be formed by removing $z \ge 0$ triples from B_2'; if $z \ge 1$, choose one that contains the difference 3. So $|B_2| = |B_2'| - z = 2k - 1 - z \ge 0$. For $n = 8$, by Condition 3, $\lambda_2 = 2$ or 5. If $\lambda_2 = 2$, let $B_2 = \{\{0, 2, 5\}\}$, and if $\lambda_2 = 5$ and let $B_2 = \emptyset$. For $n = 14$, by Condition 3, $\lambda_2 = 2, 5, 8$, or 11. If $\lambda_2 = 2$, let $B_2 = \{\{0, 2, 6\}, \{0, 3, 6\}, \{0, 4, 9\}\}$. If $\lambda_2 = 5$, let $B_2 = \{\{0, 4, 9\}, \{0, 6, 13\}\}$. If $\lambda_2 = 8$, let $B_2 = \{\{0, 4, 9\}\}$. If $\lambda_2 = 11$, let $B_2 = \emptyset$. Then for $n \ge 8$, $(\mathbb{Z}_n, B_1 \cup \{b + j \mid j \in \mathbb{Z}_n, b \in B_2\})$ is the required K_3-decomposition, since if $v \in \mathbb{Z}_n \setminus \{0, 1\}$, then $d_L(v) = (2n-2) - 4 - 6((2k-1) - z) = 6z + 4 = 2\lambda_2$ and $d_L(0) = d_L(1) = (2n-2) - 6 - 6((2k-1) - z) = 2\lambda_2 - 2$. So L is evenly equitable with

$\lambda_2 n - \lambda_1$ edges. So it remains to show that L is connected. If $\lambda_2 \geq 5$ or if $n \geq 20$ and $(\lambda_2, v) \in (z, n-3)$, then since all edges of difference 3 are in L, L is connected. For $n \geq 8$, the edges of difference $1, 2$, and $n-1$ not used in B_1 form $\frac{n-2}{3}$ vertex-disjoint 3-cycles, namely $L' = \{(i, i+1, i+2) \mid i \equiv 2 \pmod 3, 2 \leq i \leq n-3\}$. If $n \geq 20$ and $(\lambda_2, v) = (2, n-2)$, then since $G_n(\{n-2\})$ has two components (induced by odd and even vertices), L is connected since it also contains $(2, 3, 4) \in L'$. If $n \in \{8, 14\}$, then L is formed from L' by adding $E(G_n(\{\frac{n}{2}\}))$, so is easily seen to be connected.

Suppose $n \equiv 4 \pmod 6$ and let $n = 6k+4$. If $n \geq 10$, then using only differences 1 and 2, let $B_1 = \{\{i, i+1, i+2\} \mid 0 \leq i \leq n-4, i \equiv 0 \pmod 3\} \cup \{\{n-2, n-1, 0\}\}$. Then each vertex has degree 2 from the triples in B_1 except vertices 0 and $n-2$ which have degree 4. Now further assume $n \geq 22$. Consider the differences in $D' = \{3, 4, ..., n-1\}$. Then $|D'| \equiv 1$ (mod 3). Let $v = n-1$ or $n-2$ if $\frac{n-4}{3} \equiv 0, 1 \pmod 4$ or $2, 3 \pmod 4$ respectively. By Corollary 2.16, there exists a set of triples B_2' that generates a K_3-decomposition of $G_n(D = D' \setminus \{v\})$. Note that $2k \geq z$, since by Condition 3, $4 + 2n(n-1) \geq 2\lambda_2 n = (6z+4)n$, so since $n-1$ is an integer and $n > 2$, $(n-1) \geq (3z+2)$, so $6k+3 \geq 3z+2$, so $2k \geq z$. Therefore, B_2 can be formed by removing $z \geq 1$ triples from B_2', one of which contains the difference 3. So $|B_2| = |B_2'| - z = 2k - z \geq 0$. For $n = 10$, by Condition 3, $\lambda_2 = 2, 5$, or 8 so assume $\lambda_2 = 5$ or 8. If $\lambda_2 = 5$, let $B_2 = \{\{0, 4, 9\}\}$, and if $\lambda_2 = 8$, let $B_2 = \emptyset$. For $n = 16$, $\lambda_2 = 2, 5, 8, 11$ or 14, so assume $\lambda_2 \in \{5, 8, 11, 14\}$. If $\lambda_2 = 5$, let $B_2 = \{\{0, 4, 15\}, \{0, 6, 13\}, \{0, 5, 14\}\}$. If $\lambda_2 = 8$, let $B_2 = \{\{0, 4, 15\}, \{0, 6, 13\}\}$. If $\lambda_2 = 11$, let $B_2 = \{\{0, 4, 15\}\}$. If $\lambda_2 = 14$, let $B_2 = \emptyset$. Then for $n \geq 10$, $(\mathbb{Z}_n, B_1 \cup \{b + j \mid j \in \mathbb{Z}_n, b \in B_2\})$ is the required K_3-decomposition, since if $v \in \mathbb{Z}_n \setminus \{0, n-2\}$, then $d_L(v) = (2n-2) - 2 - 6(2k-z) = 6z + 4 = 2\lambda_2$ and $d_L(0) = d_L(n-2) = (2n-2) - 4 - 6(2k-z) = 2\lambda_2 - 2$. So L is evenly equitable with $\lambda_2 n - \lambda_1$ edges. Since all edges of difference 3 are in L, L is connected.

Finally, if $n = 4$, Condition 3 implies $\lambda_2 = 2$ so the result follows from Theorem 2.3.

Before proceeding to the general case, because of the limitations of Theorem 2.13, three more special cases need to be handled, namely where $\lambda_1 = 3$ and $n \equiv 5 \pmod 6$ and the two cases where $n \equiv 2 \pmod 6$ and $\lambda_1 = 4$ or 6.

Case 3: Suppose $\lambda_1 = 3$ and $n \equiv 5 \pmod 6$. Conditions 1 and 2 imply that $\lambda_2 \equiv 3 \pmod 6$. First suppose $n \geq 11$. By Case 2, there exists a K_3-decomposition $(\mathbb{Z}_n, B_1)$ of $2K_n - E(L_1)$ where L_1 is a subgraph of $2K_n$ with $(\lambda_2 - 1)n - (\lambda_1 - 1)$ edges which is evenly equitable and connected. By Theorem 2.12, there exists a K_3-decomposition $(\mathbb{Z}_n, B_2)$ of $K_n - E(L_2)$ where L_2 is a subgraph of K_n consisting of an $n-1$ cycle; name the vertices so that the vertex not in the cycle has maximum degree in L_1. Then $(\mathbb{Z}_n, B_1 \cup B_2)$ is the required decomposition with $L = G[E(L_1) \cup E(L_2)]$. Now suppose $n = 5$. Then Condition 3 in conjunction with the fact that $\lambda_2 \equiv 3 \pmod 6$ implies $\lambda_2 = 3$ so the result follows by Theorem 2.3.

Case 4: Suppose $\lambda_1 = 4$ and $n \equiv 2 \pmod 6$ with $n > 2$. Conditions 1 and 2 imply that $\lambda_2 \equiv 1 \pmod 3$. By Case 2, there exist K_3-decompositions $(\mathbb{Z}_n, B_1)$ and $(\mathbb{Z}_n, B_2)$ of $2K_n - E(L_1)$ and $2K_n - E(L_2)$ respectively where L_1 is a subgraph of $2K_n$ with $(\lambda_2 - 2)n - (\lambda_1 - 2)$ edges which is evenly equitable and connected, and L_2 is a subgraph of $2K_n$ with $2n - 2$ edges which is evenly equitable and connected. Name the vertices so that $G[E(L_1) \cup E(L_2)]$ is also evenly equitable. Then $(\mathbb{Z}_n, B_1 \cup B_2)$ is the required decomposition with $L = G[E(L_1) \cup E(L_2)]$.

Case 5: Suppose $\lambda_1 = 6$ and $n \equiv 2 \pmod 6$ with $n > 2$. Conditions 1 and 2 imply that $\lambda_2 \equiv 0 \pmod 3$. By Cases 4 and 2, there exist K_3-decompositions $(\mathbb{Z}_n, B_1)$ and $(\mathbb{Z}_n, B_2)$ of $4K_n - E(L_1)$ and $2K_n - E(L_2)$ respectively where L_1 is a subgraph of $4K_n$ with $(\lambda_2 - 2)n - (\lambda_1 - 2)$ edges which is evenly equitable and connected, and L_2 is a subgraph of $2K_n$ with $2n - 2$ edges which is evenly equitable and connected. Name the vertices so that $G[E(L_1) \cup E(L_2)]$ is also evenly equitable. Then $(\mathbb{Z}_n, B_1 \cup B_2)$ is the required decomposition with $L = G[E(L_1) \cup E(L_2)]$.

To this point, the theorem is proved if $\lambda_1 \in \{1,2\}$ and if $(\lambda_1, n) \in \{(3, 5 \pmod 6), (4, 2 \pmod 6), (6, 2 \pmod 6)\}$. So now consider the remaining cases. A two-step approach is taken in each of three cases, making use of Theorems 2.13 and 2.17.

Case 1: Suppose $n \equiv 0, 1, 3$, or $4 \pmod 6$. Let $\lambda' \in \{1,2\}$ with $\lambda' \equiv n \pmod 2$. Set $\lambda^* = \lambda_1 - \lambda'$. Since either n or $n-1$ is divisible by 3 in this case, $\lambda_1 \frac{n(n-1)}{2} \equiv 0 \pmod 3$. Also, $\lambda_1 \frac{n(n-1)}{2} - (\lambda_2 n - \lambda_1) = \lambda_1(1 + \frac{n(n-1)}{2}) - \lambda_2 n = \lambda_1(\frac{m(m-1)}{2} + \frac{n(n-1)}{2}) - \lambda_2 n \equiv \lambda_1(\frac{m(m-1)}{2} + \frac{n(n-1)}{2}) + 2\lambda_2 n \pmod 3 \equiv 0 \pmod 3$ where the last step follows from Condition 1. Therefore, $\lambda_2 n - \lambda_1$ is divisible by 3 . By Condition 2, $\lambda_2 n - \lambda_1$ is even. Since $n \equiv 0, 1, 3$ or $4 \pmod 6$, $\lambda' \frac{n(n-1)}{2}$ is divisible by 3.

First, suppose $\lambda_2 n - \lambda_1 \leq n$. By Theorem 2.13, there exists a packing (V, B_1) of $\lambda' K_n$ with leave a cycle on $\lambda_2 n - \lambda_1$ vertices. By Condition 1 and by definition, both λ_1 and λ' respectively are even when n is even; hence λ^* is even when n is even. Since $n \equiv 0, 1 \pmod 3$ and λ^* is even when n is even, by Theorem 2.3, there exists a K_3-decomposition (V, B_2) of $\lambda^* K_n$. Then $(V, B_1 \cup B_2)$ is the required packing.

Now suppose $\lambda_2 n - \lambda_1 > n$. By Theorem 2.13, let (V, B_1) be a packing of $\lambda' K_n$ with leave a cycle on $\epsilon = n$ vertices when $n \equiv 0 \pmod 3$ and $\epsilon = n - 1$ vertices when $n \equiv 1 \pmod 3$. Again λ^* is even if n is even so $\lfloor \frac{\lambda^*(n-1)}{2} \rfloor = \frac{\lambda^*(n-1)}{2}$. Further, since in this case either n or $n-1$ is divisible by 3, $\lfloor \frac{n}{3} \frac{\lambda^*(n-1)}{2} \rfloor = \frac{n}{3} \frac{\lambda^*(n-1)}{2}$. So by Theorem 2.17, there exists an evenly equitable partial triple system (V, B_2) of $\lambda^* K_n$ with $\frac{\lambda^* \frac{n(n-1)}{2} - (\lambda_2 n - \lambda_1 - \epsilon)}{3}$ triples; this is an integral number of triples since each of $\lambda^* \frac{n(n-1)}{2}$, $\lambda_2 n - \lambda_1$, and ϵ is divisible by 3. Finally, if $n \equiv 1 \pmod 3$, then name the symbols in (V, B_2) so that the vertex left out of the $n-1$ cycle gets maximum degree in the leave of (V, B_2). Then $(V, B_1 \cup B_2)$ is the required packing since $|E(L)| = \epsilon + (\lambda^* \frac{n(n-1)}{2} - 3(\frac{\lambda^* \frac{n(n-1)}{2} - (\lambda_2 n - \lambda_1 - \epsilon)}{3})) = \lambda_2 n - \lambda_1$ and L is easily seen to be evenly equitable and connected.

Case 2: Suppose $n \equiv 5 \pmod 6$ and $\lambda_1 > 3$. Let $\lambda' \in \{1,2,3\}$ with $\lambda' \equiv \lambda_1 \pmod 3$. Let $\lambda^* = \lambda_1 - \lambda'$. By the same argument as when $n \equiv 0$ or $1 \pmod 3$, Condition 1 implies that $\lambda_1\frac{n(n-1)}{2} - (\lambda_2 n - \lambda_1)$ is divisible by 3. Since $\lambda_1 \equiv \lambda'$ (mod 3), $\lambda'\frac{n(n-1)}{2} - (\lambda_2 n - \lambda_1)$ is also divisible by 3.

First suppose that $\lambda_2 n - \lambda_1 \leq n$. By Theorem 2.13, there exists a packing (V, B_1) of $\lambda' K_n$ with leave a cycle on $\lambda_2 n - \lambda_1$ vertices. Since $\lambda^* \equiv 0$ (mod 3), by Theorem 2.3, there exists a K_3-decomposition (V, B_2) of $\lambda^* K_n$. Then $(V, B_1 \cup B_2)$ is the required packing.

Now suppose $\lambda_2 n - \lambda_1 > n$. Since $n \equiv 5$ (mod 6), $mn = 2n \equiv 1$ (mod 3) and $\frac{m(m-1)}{2} + \frac{n(n-1)}{2} \equiv 2$ (mod 3); hence, by Condition 1, $\lambda_1 \equiv \lambda_2$ (mod 3). Since $n \equiv 2$ (mod 3) in this case, and since $\lambda_1 \equiv \lambda_2$ (mod 3), $\lambda_2 n \equiv 2\lambda_1$ (mod 3) so $\lambda_2 n - \lambda_1 \equiv \lambda_1$ (mod 3). Also, since $\frac{n(n-1)}{2} \equiv 1$ (mod 3), $\lambda'\frac{n(n-1)}{2} \equiv \lambda'$ (mod 3). By Theorem 2.13, there exists a packing (V, B_1) of $\lambda' K_n$ with leave a cycle on $\epsilon = n-1, n$, or $n-2$ vertices when $\lambda' = 1, 2$ or 3 respectively. In this case, $(n-1)$ is even and λ^* is divisible by 3, so $\lfloor \frac{n}{3} \lfloor \frac{\lambda^*(n-1)}{2} \rfloor \rfloor = \frac{n}{3}\frac{\lambda^*(n-1)}{2}$. So by Theorem 2.17, there exists an equitable partial triple system (V, B_2) of $\lambda^* K_n$ with $\frac{\lambda^*\frac{n(n-1)}{2} - (\lambda_2 n - \lambda_1 - \epsilon)}{3}$ triples; this is an integral number of triples since λ^* is divisible by 3 and $\lambda_2 n - \lambda_1 \equiv \lambda_1 \equiv \lambda' \equiv \epsilon$ (mod 3). Finally, if $\epsilon = n-1$ or $n-2$, then name the symbols of (V, B_2) so that each vertex not in the $n-1$ or $n-2$ cycle gets no smaller degree in the leave of (V, B_2) than any vertex in the cycle. Then $(V, B_1 \cup B_2)$ is the required packing, since L is clearly evenly equitable and connected and $|E(L)| = \epsilon + (\lambda^*\frac{n(n-1)}{2} - 3(\frac{\lambda^*\frac{n(n-1)}{2} - (\lambda_2 n - \lambda_1 - \epsilon)}{3})) = \lambda_2 n - \lambda_1$.

Case 3: Finally, suppose $n \equiv 2$ (mod 6). By Condition 2, λ_1 is even, so it can be assumed $\lambda_1 > 6$ (since all smaller cases were handled previously). Recall that $n > 2$. Let $\lambda' \in \{2, 4, 6\}$ with $\lambda' \equiv \lambda_1$ (mod 6). Set $\lambda^* = \lambda_1 - \lambda'$. By the same argument as when $n \equiv 0$ or 1 (mod 3), Condition 1 implies that $\lambda_1\frac{n(n-1)}{2} - (\lambda_2 n - \lambda_1)$ is divisible by 3. Since $\lambda_1 \equiv \lambda'$ (mod 3), $\lambda'\frac{n(n-1)}{2} - (\lambda_2 n - \lambda_1)$ is also divisible by 3.

First suppose that $\lambda_2 n - \lambda_1 \leq n$. By Theorem 2.13, there exists a packing (V, B_1) of $\lambda' K_n$ with leave a cycle on $\lambda_2 n - \lambda_1$ vertices. Since $\lambda^* \equiv 0$ (mod 6), by Theorem 2.3, there exists a K_3-decomposition (V, B_2) of $\lambda^* K_n$. Then $(V, B_1 \cup B_2)$ is the required packing.

Suppose $\lambda_2 n - \lambda_1 > n$. Since $n \equiv 2$ (mod 6), $mn = 2n \equiv 1$ (mod 3) and $\frac{m(m-1)}{2} + \frac{n(n-1)}{2} \equiv 2$ (mod 3); hence, by Condition 1, $\lambda_1 \equiv \lambda_2$ (mod 3). Since $n \equiv 2$ (mod 3) in this case, and since $\lambda_1 \equiv \lambda_2 \pmod 3$, $\lambda_2 n \equiv 2\lambda_1 \pmod 3$ so $\lambda_2 n - \lambda_1 \equiv \lambda_1 \pmod 3$. Also, since $\frac{n(n-1)}{2} \equiv 1 \pmod 3$, $\lambda' \frac{n(n-1)}{2} \equiv \lambda' \pmod 3$. By Theorem 2.13, there exists a packing (V, B_1) of $\lambda' K_n$ with leave a cycle on $\epsilon = n, n-1$, or $n-2$ vertices when $\lambda' = 2, 4$ or 6 respectively. In this case, λ^* is even and divisible by 3, so $\lfloor \frac{n}{3} \lfloor \frac{\lambda^*(n-1)}{2} \rfloor \rfloor = \frac{n}{3} \frac{\lambda^*(n-1)}{2}$. So by Theorem 2.17, there exists an equitable partial triple system (V, B_2) of $\lambda^* K_n$ with $\frac{\lambda^* \frac{n(n-1)}{2} - (\lambda_2 n - \lambda_1 - \epsilon)}{3}$ triples; this is an integral number of triples since λ^* is divisible by 3 and $\lambda_2 n - \lambda_1 \equiv \lambda_1 \equiv \lambda' \equiv \epsilon \pmod 3$. Finally, if $\epsilon = n-1$ or $n-2$, then name the symbols of (V, B_2) so that each vertex not in the $n-1$ or $n-2$ cycle gets no smaller degree in the leave of (V, B_2) than any vertex in the cycle. Then $(V, B_1 \cup B_2)$ is the required packing, since L is clearly evenly equitable and connected and $|E(L)| = \epsilon + (\lambda^* \frac{n(n-1)}{2} - 3(\frac{\lambda^* \frac{n(n-1)}{2} - (\lambda_2 n - \lambda_1 - \epsilon)}{3})) = \lambda_2 n - \lambda_1$. $\square$

Chapter 3

Neighborhoods in Maximum Packings of $2K_n$ and Quadratic Leaves of Triple Systems

3.1 Introduction

In this chapter, the quadratic leave problem is considered again, this time with the focus of allowing 2-cycles in the leave. To do this, a characterization of the possible neighborhood graphs in maximum packings of $2K_n$ is given which in turn will be used to solve the quadratic leave problem. As mentioned in the introduction, in [9], Colbourn and Rosa characterized the possible neighborhood graphs in a 2-fold triple system on n vertices when $n \equiv 0$ or 1 (mod 3). (A 2-fold triple system is equivalent to a maximum packing in these two cases.) In this chapter, a characterization of the possible neighborhood graphs of vertices in a maximum packing of $2K_n$ for $n \equiv 2 \pmod 3$ is given (see Theorem 3.9). When $n \equiv 2 \pmod 3$ the leave of a maximum packing (V, B) of $2K_n$ is the 2-cycle (a, b) for some $a, b \in V$ (with $a \neq b$); see Lemma 3.2. So in this case an additional interesting aspect of finding the neighborhood of a vertex v in a maximum packing of $2K_n$ arises; the neighborhood is a 2-regular graph on $n - 2$ vertices if $v \in \{a, b\}$ and is a 2-regular graph on $n - 1$ vertices otherwise. The proof technique in this chapter builds on modern observations recently made independently in two papers [3, 22] concerning the existence of leaves of partial hamilton cycle decompositions of K_n.

Bryant, Horsley, and Maenhaut have some results which relate to Theorem 3.10. In [4], they show that K_n can be decomposed into 2-regular subgraphs of orders $m_1, \ldots, m_t$ provided that n is odd, $3 \leq m_i \leq n$ for $1 \leq i \leq t$, and $\sum m_i = \binom{n}{2}$. Note that specifying $m_1, m_2, \ldots, m_j$ $(j < t)$ to be the lengths of the cycles described in Theorem 3.10 is not sufficient to force the 2-regular subgraphs to be cycles, nor to force the cycles in the quadratic graph to be vertex disjoint. In [6], Bryant and Horsley extend their first result by showing that K_n can be decomposed into cycles of orders $m_1, \ldots, m_t$ provided that n is odd, $3 \leq m_i \leq n$ for $1 \leq i \leq t$, and $\sum m_i = \binom{n}{2}$ whenever n is sufficiently large. Note that Theorem 3.10 cannot be obtained from this result in the special case where $\lambda = 1$ and n is large enough since the cycles in the quadratic graph are not forced to be vertex disjoint.

Maximal cycle systems have also been of interest from another perspective. Rather than study the structure of the leave, several papers have considered its size, addressing the spectrum question of finding the set S of integers for which there exists a partial k-cycle system with leave of size l for each $l \in S$. Cycles of length 3 and hamilton cycles have been of particular interest (see for instance [12, 8, 28]).

For any multiset D with elements chosen from $\{1, 2, \ldots, n-1\}$, let $G_n(\{D\})$ be the multigraph with vertex set $\mathbb{Z}_n$ and edge multiset $\{\{v, v+d\} \mid d \in D, v \in \mathbb{Z}_n\}$ reducing sums modulo n. Note that with this definition $G_n(\{d\}) = G_n(\{n-d\})$ and is a 2-regular graph regardless of the value of d (if $d = \frac{n}{2}$ then each component is a 2-cycle). The edges in $G_n(\{d\})$ are said to have difference d. This definition is slightly non-standard since edges are allowed to have difference $d > \frac{n}{2}$, but this approach is very useful when decomposing $2K_n$.

If $\{a, b, c\}$ is a triple on the vertex set $\mathbb{Z}_n$ then define $\{a, b, c\} + j = \{a+j, b+j, c+j\}$, reducing sums modulo n. Similarly, if $(a_0, a_1, \ldots, a_n)$ is a cycle on the vertex set $\mathbb{Z}_n$ then define $(a_0, a_1, \ldots, a_n) + j = (a_0 + j, a_1 + j, \ldots, a_n + j)$, reducing sums modulo n.

Throughout the chapter, λ and n are assumed to be positive integers.

3.2 Preliminary Results

To start, a well-known result on quasigroups is given.

Lemma 3.1. [27] *There exists an idempotent quasigroup of order n for all $n \neq 2$.*

The following is a specific case of well-known results on packings of λK_n, and is sufficient for the purposes of this chapter.

Lemma 3.2. [14] *Suppose $\lambda \geq 1$ and $n \neq 2$. There exists a K_3-decomposition of λK_n if and only if λ is divisible by*

1. *2 if $n \equiv 0$ or 4 (mod 6),*
2. *3 if $n \equiv 5$ (mod 6), and*
3. *6 if $n \equiv 2$ (mod 6).*

Furthermore, there exists a maximum packing of λK_n with leave L where:

1. *L is a 4-cycle if $\lambda = 4$ and $n \equiv 2$ (mod 3) or $\lambda = 1$ and $n \equiv 5$ (mod 6), and*
2. *L is a 2-cycle if $\lambda = 2$ and $n \equiv 2$ (mod 3).*

The next result is a neat way to be able to deal with the many possibilities for Q.

Lemma 3.3. *For any quadratic multigraph Q on n vertices, there exists a subgraph of $G_n(\{1,1,2\})$ which is isomorphic to Q.*

Proof. For each l with $2 \leq l \leq n$, define the cycle $c(l)$ of length l in $G_n(\{1,1,2\})$ as follows. If $l = 2$ then let $c(l) = (0,1)$. If $l = 2x > 2$ then let $c(l) = (0,2,4,6,\ldots,2x-2,2x-1,2x-3,2x-5,\ldots,1)$. Finally, if $l = 2x-1$ then let $c(l) = (0,2,\ldots,2x-2,2x-3,2x-5,\ldots,1)$. Suppose that Q is the disjoint union of t cycles of lengths $l_1,\ldots,l_t$ (possibly $l_i = 2$). For $1 \leq i \leq t$ let $s_i = \sum_{j=1}^{i} l_j$, and let $c_i = c(l_i) + s_{i-1}$, defining $s_0 = 0$. Then $\bigcup_{i=1}^{t} c_i$ is a subgraph of $G_n(\{1,1,2\})$ and is isomorphic to Q. □

The following result was proved by Petersen.

Lemma 3.4. [26] *Let H be any $2k$-regular multigraph. There exists a 2-factorization of H.*

In constructing maximum packings, it will always be first assumed that the desired neighborhood contains a small cycle, the size of which will depend upon the current case. The following lemma is a well-known approach that will allow for the extension of these constructions to cases where the prescribed small cycle is not present by combining two cycles in the neighborhood of a vertex v into a single cycle.

Lemma 3.5. *Let (V, S) be a partial 2-fold triple system, and let $v \in V$. Suppose that $\{a, b\}$ and $\{c, e\}$ are edges in disjoint cycles in the neighborhood of v, and further suppose that there is some $w \in V$ with $w \neq v$ such that $\{\{a, c, w\}, \{b, e, w\}\} \subset S$. Then there exists a partial 2-fold triple system (V, S') such that*

1. *$E(S) = E(S')$,*
2. *the neighborhood of v in S' is obtained from the neighborhood of v in S by replacing one copy of the edges $\{a, b\}$ and $\{c, e\}$ with one copy of the edges $\{a, c\}$ and $\{b, e\}$.*

Proof. Define $S' = (S \setminus \{\{v, a, b\}, \{v, c, e\}, \{w, a, c\}, \{w, b, e\}\})$
$\cup\{\{v, a, c\}, \{v, b, e\}, \{w, a, b\}, \{w, c, e\}\}$. □

The next lemma settles cases that will be used in Section 3 for the general construction, and includes a proof of Theorem 3.9 when $n \in \{5, 8\}$. When $n \in \{4, 6\}$, the results follow from the results of Colbourn and Rosa but are included here for completeness.

Lemma 3.6. *Let $S = \{(4, C_3), (6, C_5), (5, C_3), (5, C_4), (8, C_3 \cup C_3), (8, C_2 \cup C_4), (8, C_2 \cup C_2 \cup C_2), (8, C_6), (8, C_4 \cup C_3), (8, C_2 \cup C_5), (8, C_2 \cup C_2 \cup C_3), (8, C_7)\}$. For each $(n, Q) \in S$, there exists a maximum packing of $2K_n$ such that the neighborhood of some vertex is Q.*

Proof. Let the vertex set of K_n be $V = \{a, b\} \cup \mathbb{Z}_{n-2}$. In each case, a maximum packing is formed with leave the 2-cycle (a, b) if $n \in \{5, 8\}$ and the empty leave if $n \in \{4, 6\}$. All addition in the following is defined modulo $n - 2$. In each case, the set of blocks defined produce a maximum packing of $2K_n$.

Let $n = 4$. Define $B = \{\{a, b, 0\}, \{a, b, 1\}, \{a, 0, 1\}, \{b, 0, 1\}\}$. Then the neighborhood of a is C_3.

Let $n = 6$. Define $B = \{\{a, b, 0\}, \{a, b, 1\}, \{a, 0, 2\}, \{a, 1, 3\}, \{a, 2, 3\}, \{b, 0, 3\}, \{b, 1, 2\}, \{b, 2, 3\}, \{0, 1, 2\}, \{0, 1, 3\}\}$. Then the neighborhood of a is C_5.

Let $n = 5$. Define $B = \{\{j, i, i + 1\} \mid j \in \{a, b\}, 0 \leq i \leq 2\}$. Then the neighborhood of a is C_3 and the neighborhood of 0 is C_4.

Let $n = 8$. Define $B_1 = \{\{i, i+1, i+3\}, \{a, i, i+2\}, \{b, i, i+1\} \mid i \in \mathbb{Z}_6\}$, $B_2 = \{\{i, i+1, i+2\}, \{a, i, i+3\}, \{b, i, i+2\} \mid i \in \mathbb{Z}_6\}$, $B_3 = \{\{0,2,4\}, \{0,2,5\}, \{0,3,5\}, \{1,2,4\}, \{1,3,4\}, \{1,3,5\}, \{b,0,3\}, \{b,0,4\}, \{b,1,2\}, \{b,1,5\}, \{b,2,3\}, \{b,4,5\}, \{a,0,1\}, \{a,0,1\}, \{a,2,3\}, \{a,3,4\}, \{a,4,5\}, \{a,2,5\}\}$, and $B_4 = \{\{a,0,1\}, \{a,0,1\}, \{0,2,3\}, \{0,2,3\}, \{0,4,5\}, \{0,4,b\}, \{0,5,b\}, \{a,2,4\}, \{a,2,5\}, \{a,3,4\}, \{a,3,5\}, \{1,2,4\}, \{1,2,b\}, \{1,3,5\}, \{1,3,b\}, \{1,4,5\}, \{2,5,b\}, \{3,4,b\}\}$. In B_1, the neighborhood of a is $C_3 \cup C_3$ and of b is C_6. In B_2, the neighborhood of a is $C_2 \cup C_2 \cup C_2$. In B_3, the neighborhood of a is $C_2 \cup C_4$, of 0 is $C_2 \cup C_5$, and of 2 is C_7. In B_4, the neighborhood of 0 is $C_2 \cup C_2 \cup C_3$. Finally, using an approach similar to Lemma 3.5, the neighborhood of 0 in $(B_4 \setminus \{\{0,1,a\}, \{0,2,3\}\{a,2,5\}, \{1,3,5\}\}) \cup \{\{a,0,2\}, \{0,1,3\}, \{a,1,5\}, \{2,3,5\}\}$ is $C_4 \cup C_3$. $\square$

Before proceeding to the main theorem of this chapter, the powerful result on Langford sequences that was proved over a series of papers is given again (see for example [29]).

A Langford sequence of order $m \geq 1$ and defect $\delta \geq 1$ is a sequence $L = (l_1, l_2, \ldots, l_{2m})$ of $2m$ positive integers satisfying the conditions

(a) for every $k \in \{\delta, \delta+1, \ldots, \delta+m-1\}$, there exist exactly two integers l_i, l_j in L such that $l_i = l_j = k$, and

(b) if $l_i = l_j = k$, then $|i - j| = k$.

A hooked Langford sequence of order $m \geq 1$ and defect $\delta \geq 1$ is a sequence $L = (l_1, l_2, \ldots, l_{2m}, l_{2m+1})$ of $2m+1$ nonnegative integers satisfying the conditions

(a) $l_{2m} = 0$

(b) for every $k \in \{\delta, \delta+1, \ldots, \delta+m-1\}$, there exist exactly two integers l_i, l_j in L such that $l_i = l_j = k$, and

(c) if $l_i = l_j = k$, then $|i - j| = k$.

For emphasis, a Langford sequence is sometimes called a perfect Langford sequence.

Theorem 3.7. [29] *A Langford sequence of order m and defect δ exists if and only if*

1. *$m \geq \delta - 1$, and*

2. *either $m \equiv 0, 1$ (mod 4) and δ is odd, or $m \equiv 0, 3$ (mod 4) and δ is even.*

A hooked Langford sequence of order m and defect δ exists if and only if

3. *$m(m - 2\delta + 1) + 2 \geq 0$, and*

4. *either $m \equiv 2, 3$ (mod 4) and δ is odd, or $m \equiv 1, 2$ (mod 4) and δ is even.*

Remark: Notice that every pair of integers m and δ satisfies either Condition 2 or Condition 4. Also, if $\delta = 2$ then Conditions 1 and 3 are satisfied for all $m \geq 1$ when Conditions 2 and 4 respectively are satisfied.

The following well-known result is the purpose of introducing these sequences.

Lemma 3.8. *If there exists a Langford sequence or a hooked Langford sequence of order m and defect δ then, for each $n > \delta + 3m$, there exists a K_3-decomposition of $G_n(\{\delta, \delta + 1, \ldots, \delta + 3m - 1\})$ or of $G_n(\{\delta, \delta + 1, \ldots, \delta + 3m - 2, \delta + 3m\})$ respectively.*

Proof. Let $(l_1, \ldots, l_{2m})$ or $(l_1, \ldots, l_{2m+1})$ be a Langford sequence or a hooked Langford sequence respectively. Define $B = \{\{0, \delta + m - 1 + i, \delta + m - 1 + j\} + t \mid l_i = l_j, t \in \mathbb{Z}_n\}$. Then $(\mathbb{Z}_n, B)$ is the required decomposition. $\square$

3.3 Neighborhoods for $2K_{3x+2}$

The first of the two main results of this chapter is now stated and proved.

Theorem 3.9. *Let $n \equiv 2$ (mod 3) with $n > 2$, and let Q be a 2-regular multigraph on either $n - 2$ or $n - 1$ vertices. Then there exists a maximum packing of $2K_n$ with leave a 2-cycle such that the neighborhood graph of some vertex is Q if and only if $(n, Q) \neq (5, C_2 \cup C_2)$.*

Proof. If there exists a maximum packing of $2K_5$ such that the neighborhood of some vertex is $C_2 \cup C_2$, then deleting the triples containing this vertex leaves the graph $2K_{2,2}$ which contains no K_3; so this case is not possible.

The sufficiency is now proved. So let Q be a 2-regular graph on $n-2$ or $n-1$ vertices, $n \equiv 2 \pmod 3$, and $(n, Q) \neq (5, C_2 \cup C_2)$. Let $|V(Q)| = n - 2 + \epsilon$ with $\epsilon \in \{0, 1\}$. In each case, a maximum packing (V, B) of $2K_n$ is produced in which the neighborhood of the vertex ∞_0 is Q. If $n \in \{5, 8\}$ then the result follows from Lemma 3.6, so it can be assumed that $n = 3k + 5 \geq 11$.

Note that if $|V(Q)| = n - 2$ then ∞_0 must be in the leave, so the maximum packing will be constructed with leave the 2-cycle (∞_0, ∞_1). If $|V(Q)| = n - 1$ then ∞_0 cannot be in the leave, so for notational convenience the maximum packing will be constructed with leave the 2-cycle (∞_1, ∞_3).

Case 1: Suppose that Q contains a cycle c of length $3 + \epsilon$. Let $V = \bigcup_{i=0}^{4}\{\infty_i\} \cup \mathbb{Z}_{3k}$ and name the cycle $c = (\infty_{2-\epsilon}, \infty_{3-\epsilon}, \ldots, \infty_4)$. By Lemma 3.6, let $(\{\infty_i \mid i \in \mathbb{Z}_5\}, B_0)$ be a maximum packing of $2K_5$ with leave the 2-cycle $(\infty_1, \infty_{3\epsilon})$ in which the neighborhood of ∞_0 is the $(3 + \epsilon)$-cycle c.

By Lemma 3.3, let G_0 be a subgraph of $G_{3k}(\{1, 3k-2, 3k-1\})$ isomorphic to $Q \setminus \{c\}$. Since $G_{3k}(\{1, 3k-2, 3k-1\}) - E(G_0)$ is a 4-regular graph, by Lemma 3.4 it has a 2-factorization $\{G_1, G_2\}$. Let $d = 3k - 4$ if $k - 2 \equiv 0$ or $3 \pmod 4$ and let $d = 3k - 5$ if $k - 2 \equiv 1$ or $2 \pmod 4$. By Lemma 3.4 let $\{G_3, G_4\}$ be a 2-factorization of $G_{3k}(\{3k-3, d\})$. Let $B_1 = \{\{\infty_i, y, z\} \mid \{y, z\} \in E(G_i), 0 \leq i \leq 4\}$. So the neighborhood of ∞_0 is Q, and all that remains to do is to partition the edges of $G_{3k}(\{2, 3, \ldots, 3k-4\} \setminus \{d\})$ into triples; note that if $k = 2$ then this graph has no edges, so it can be assumed that $k \geq 3$. By the remark following Theorem 3.7, there exists either a perfect or a hooked Langford sequence of order $k - 2 \geq 1$ and defect 2, so the choice of d ensures that by Corollary 3.8 there exists a K_3-decomposition $(\mathbb{Z}_{3k}, B_2)$ of $G_{3k}(\{2, 3, \ldots, 3k-4\} \setminus \{d\})$. Then $(\{\infty_i \mid i \in \mathbb{Z}_5\} \cup \mathbb{Z}_{3k}, \bigcup_{i=0}^{2} B_i)$ is the required decomposition.

Case 2: Suppose that Q contains a cycle c_0 of length $x+4+\epsilon \geq 5+\epsilon$. Let $V = \bigcup_{i=0}^{4}\{\infty_i\} \cup \mathbb{Z}_{3k}$ and name the cycle $c_0 = (0,1,\ldots,x-1,\infty_{2-\epsilon},\infty_{3-\epsilon},\ \ldots,\infty_4,x)$. Define the cycles $c_0' = (\infty_{2-\epsilon},\infty_{3-\epsilon},\ldots,\infty_4)$ and $c_0'' = (0,1,2,\ldots,x-1,x)$ (so if $x=1$ then c_0'' is a 2-cycle). Let Q' be formed from Q by replacing c_0 with $c_0' \cup c_0''$. Since Q' contains a cycle of length exactly $3+\epsilon$, the argument in Case 1 can be used to produce a packing (V,B_0) of $2K_{n-5}$ with leave $\bigcup_{i=0}^{4} G_i$ where $G_0 = Q' \setminus \{c_0'\}$ and $\{G_1,\ldots,G_4\}$ is a 2-factorization of $G' = G_{3k}(\{1,3k-1,3k-2,3k-3,d\}) - E(G_0)$ with $d \in \{3k-4,3k-5\}$. Note that $\{x-1,x\} \subset V(c_0'')$ and that $x+1 \notin V(c_0'')$ since $x+4+\epsilon \leq |V(Q)| = n-2+\epsilon = 3k+2+\epsilon$ implies that $x \leq 3k-2$ (so $x+1 \neq 0$). Therefore G' contains two copies of the edge $\{x,x+1\}$ (one of difference 1 and one of difference $3k-1$) and one copy of the edge $\{x-1,x+1\}$ (of difference $3k-2$), so one copy of the edge $\{x,x+1\}$ must be in a different 2-factor than the edge $\{x-1,x+1\}$; say these 2-factors are G_4 and $G_{2-\epsilon}$ respectively. By Lemma 3.6, let $(\{\infty_i \mid i \in \mathbb{Z}_5\},B_1)$ be a maximum packing of $2K_5$ with leave the 2-cycle $(\infty_1,\infty_{3\epsilon})$ such that the neighborhood of ∞_0 is the $(\epsilon+3)$-cycle c_0'. Then $(\{\infty_i \mid i \in \mathbb{Z}_5\} \cup \mathbb{Z}_{3k}, B_0 \cup B_1 \cup \{\{\infty_i,a,b\} \mid \{a,b\} \in E(G_i)\})$ is a maximum packing of $2K_n$ such that the neighborhood of ∞_0 is Q'. Finally, by applying Lemma 3.5 to this maximum packing with $v=\infty_0$, $w=x+1$, $a=\infty_{2-\epsilon}$, $b=\infty_4$, $c=x-1$, and $e=x$, a maximum packing of $2K_n$ in which the neighborhood of ∞_0 is Q is produced.

Case 3: Suppose that $\epsilon=1$ and each cycle in Q is a 2-cycle. Then n is odd so let $n = 6l+5$ with $l \geq 1$. Let $V = \{\infty_0\} \cup (\mathbb{Z}_{3l+2} \times \mathbb{Z}_2)$. By Lemma 3.1, let $(\mathbb{Z}_{3l+2},\circ)$ be an idempotent quasigroup. By Lemma 3.2, there exists a maximum packing $(\mathbb{Z}_{3l+2} \times \{1\}, B_1)$ of $2K_{3l+2}$ with leave a 2-cycle c. Then $(\{\infty_0\} \cup \mathbb{Z}_{3l+2} \times \mathbb{Z}_2, B_1 \cup \{\{(a,0),(b,0),(a \circ b,1)\},\{(a,0),(b,0),(b \circ a,1)\} \mid 0 \leq a < b \leq 3l+1\} \cup \{\{\infty_0,(a,0),(a,1)\},\{\infty_0,(a,0),(a,1)\} \mid a \in \mathbb{Z}_{3l+2}\})$ is the required decomposition with leave c.

Case 4: In view of Cases 1 and 2, suppose that if $\epsilon=0$ then each cycle in Q has length 2 or 4, and if $\epsilon=1$ then each cycle in Q has length $2,3$, or 5. In view of Case 3, if $\epsilon=1$ then also assume that Q contains at least one cycle of length 3 or 5.

Case 4.1: Assume that $n \geq 17$. In this subcase it is convenient to redefine k so that $n = 3k + 8$; so $k \geq 3$. Let $V = \{\infty_i \mid i \in \mathbb{Z}_8\} \cup \mathbb{Z}_{3k}$.

Suppose $\epsilon = 0$. If Q contains only 4-cycles then let Q' be formed from Q be replacing one 4-cycle, say $(1, \infty_6, \infty_7, 2)$, with the two 2-cycles (∞_6, ∞_7) and $(1, 2)$, and if Q contains a 2-cycle, say (∞_6, ∞_7), then let $Q' = Q$. We can assume that either Q' contains both the 4-cycle $c_1 = (\infty_2, \infty_3, \infty_4, \infty_5)$ and the 2-cycle $c_2 = (\infty_6, \infty_7)$, or Q' contains the three 2-cycles $c_1 = (\infty_2, \infty_3), c_2 = (\infty_4, \infty_5)$, and $c_3 = (\infty_6, \infty_7)$; let $c^* = \{c_1, c_2\}$ or $\{c_1, c_2, c_3\}$ respectively.

Suppose $\epsilon = 1$ (and hence $|V(Q)| = n - 1 \equiv 1 \pmod 3$). Since $|V(Q)| \equiv 1$ (mod 3), Q contains at least two cycles of length 2 (mod 3). If Q contains two 5-cycles then let them be $(0, 1, \infty_6, \infty_7, 2)$ and $c_1 = (\infty_1, \infty_2, \infty_3, \infty_4, \infty_5)$; form Q' from Q be replacing the first 5-cycle with the two cycles $c_2 = (\infty_6, \infty_7)$ and $c_3 = (0, 1, 2)$, and let $c^* = \{c_1, c_2\}$. If Q contains exactly one 5-cycle and a 2-cycle then let them be $c_1 = (\infty_1, \infty_2, \infty_3, \infty_4, \infty_5)$ and $c_2 = (\infty_6, \infty_7)$; let $Q' = Q$ and $c^* = \{c_1, c_2\}$. If Q contains no 5-cycles then it must contain two 2-cycles and a 3-cycle (in view of Case 3), say $c_1 = (\infty_1, \infty_2)$, $c_2 = (\infty_6, \infty_7)$, and $c_3 = (\infty_3, \infty_4, \infty_5)$; let $Q' = Q$ and $c^* = \{c_1, c_2, c_3\}$.

In either case, by Lemma 3.6 let $(\{\infty_i \mid i \in \mathbb{Z}_8\}, B_0)$ be a maximum packing of $2K_8$ with leave the 2-cycle $(\infty_1, \infty_{3\epsilon})$ such that the neighborhood of ∞_0 is c^*.

By Lemma 3.3, let G_0 be a subgraph of $G_{3k}(\{1, 3k-2, 3k-1\})$ isomorphic to $Q' \setminus \{c^*\}$. Let $d = 3k - 7$ if $k - 3 \equiv 0$ or $3 \pmod 4$ and let $d = 3k - 8$ if $k - 3 \equiv 1$ or $2 \pmod 4$. Since $G = G_{3k}(\{1, 3k-1, 3k-2, 3k-3, 3k-4, 3k-5, 3k-6, d\}) - E(G_0)$ is a 14-regular graph, by Lemma 3.4 it has a 2-factorization $\{G_1, G_2, \ldots, G_7\}$. Note that if $Q' \neq Q$, then $Q' \setminus c^*$ contains either the cycle $(1, 2)$ or the cycle $(0, 1, 2)$, which implies that G_0 does not contain any copies of the edge $\{1, 3\}$ or $\{2, 3\}$ and hence that G contains two copies of the edge $\{2, 3\}$ and one copy of the edge $\{1, 3\}$. Thus one copy of the edge $\{2, 3\}$ must be in a different 2-factor than the edge $\{1, 3\}$; say these 2-factors are G_7 and G_6 respectively. Let $B_1 = \{\{\infty_i, y, z\} \mid \{y, z\} \in E(G_i), 0 \leq i \leq 7\}$. So the neighborhood of ∞_0 is Q', and all that remains to do is to partition the edges of $G_{3k}(\{2,3,\ldots,3k-7\}\setminus\{d\})$ into triples; to do so, note that this graph is graph is empty if $k=3$, so we can assume that $k\geq 4$. There exists either a perfect or a hooked Langford sequence of order $k-3\geq 1$ and defect 2, so the choice of d ensures that by Corollary 3.8 there exists a K_3-decomposition $(\mathbb{Z}_{3k},B_2)$ of $G_{3k}(\{2,3,\ldots,3k-7\}\setminus\{d\})$. Then $(\{\infty_i\mid i\in\mathbb{Z}_8\}\cup\mathbb{Z}_{3k},\bigcup_{i=0}^{2}B_i)$ is a maximum packing of $2K_n$ such that the neighborhood of ∞_0 is Q'. If $Q'\neq Q$ then apply Lemma 3.5 with $v=\infty_0$, $w=3$, $a=\infty_6$, $b=\infty_7$, $c=1$, and $e=2$ to produce a maximum packing of $2K_n$ in which the neighborhood of ∞_0 is Q.

Case 4.2: Assume that $n=14$, $|V(Q)|=13$, and Q contains a 5-cycle, c. By Lemma 3.6, let $(\{\infty_i\mid i\in\mathbb{Z}_6\},B_1)$ be a K_3-decomposition of $2K_6$ such that the neighborhood of ∞_0 is the 5-cycle $c\in Q$. By Lemma 3.3, let G_0 be a subgraph of $G_8(\{1,6,7\})$ isomorphic to $Q\setminus\{c\}$. Since $G_8(\{1,6,7\})-E(G_0)$ is a 4-regular graph, by Lemma 3.4 it has a 2-factorization $\{G_1,G_2\}$. Let $B_2=\{\{0,2,5\},\{1,3,6\}\}$. Then $(G_8(\{2,3,4,5\}))\setminus(E(B_2)\cup\{4,7\}\cup\{4,7\})$ is a 6-regular graph so it has a 2-factorization $\{G_3,G_4,G_5\}$. Then $(\{\infty_i\mid i\in\mathbb{Z}_6\}\cup\mathbb{Z}_8,B_1\cup B_2\cup\{\{a,b,\infty_i\}\mid\{a,b\}\in E(G_i),0\leq i\leq 5\})$ is the required decomposition (with leave the 2-cycle $(4,7)$).

Case 4.3: Assume that $n=14$, $|V(Q)|=13$, and Q contains a 3-cycle, c. By Lemma 3.6, let $(\{\infty_i\mid i\in\mathbb{Z}_4\},B_1)$ be a K_3-decomposition of $2K_4$ such that the neighborhood of ∞_0 is the 3-cycle $c\in Q$. By Lemma 3.3, let G_0 be a subgraph of $G_{10}(\{1,8,9\})$ isomorphic to $Q\setminus\{c\}$. Since $G_{10}(\{1,8,9\})-E(G_0)$ is a 4-regular graph, by Lemma 3.4 it has a 2-factorization $\{G_1,G_2\}$. Let $B_2=\{\{0,2,5\},\{0,5,7\},\{1,3,6\},\{1,6,8\},\{2,4,7\},\{2,7,9\},\{3,5,8\},\{0,3,8\},\{0,4,6\},\{1,5,9\},\{0,3,7\},\{1,4,7\},\{1,4,8\},\{3,6,9\},\{2,6,9\},\{2,5,8\}\}\cup\{\{\infty_3,i,i+4\}\mid i\in\mathbb{Z}_{10}\}$. Then $(\{\infty_i\mid i\in\mathbb{Z}_4\}\cup\mathbb{Z}_{10},B_1\cup B_2\cup\{\{a,b,\infty_i\}\mid\{a,b\}\in E(G_i),0\leq i\leq 2\})$ is the required decomposition (with leave the 2-cycle $(4,9)$).

Case 4.4: Assume that $n=14$, $|V(Q)|=12$, and Q contains α 4-cycles where $0\leq\alpha\leq 3$. Let $B=(\{\{i,i+3,i+4\},\{i,i+2,i+5\},\{i,i+4,i+5\},\{\infty_0,i,i+6\},\{\infty_1,i,i+2\}\mid i\in\mathbb{Z}_{12}\}\setminus\{\{j,j+3,j+4\},\{j+4,j+6,j+9\},\{\infty_0,j,j+6\},\{\infty_0,j+3,j+9\}\mid 1\leq j\leq$

$\alpha\}) \cup \{\{j, j+4, j+6\}, \{j+3, j+4, j+9\}, \{\infty_0, j, j+3\}, \{\infty_0, j+6, j+9\} \mid 1 \leq j \leq \alpha\}$. Then $(\mathbb{Z}_{12} \cup \{\infty_0, \infty_1\}, B)$ is a maximum packing of $2K_{14}$ with leave (∞_0, ∞_1) such that the neighborhood of ∞_0 consists of α 4-cycles and $\frac{12-4\alpha}{2}$ 2-cycles as required.

Case 4.5: Assume that $n = 11$. It is impossible for $|Q| = 9$ and to have Q consist only of 2-cycles and 4-cycles, so assume that $|Q| = 10$. Let $B = \{\{\infty_0, 0, 1\}, \{\infty_0, 0, 1\}, \{\infty_0, 2, 3\}, \{\infty_0, 2, 3\}, \{\infty_0, 4, 5\}, \{\infty_0, 4, 6\}, \{\infty_0, 5, 6\}, \{\infty_0, 7, 8\}, \{\infty_0, 7, 9\}, \{\infty_0, 8, 9\}, \{0, 2, 4\}, \{0, 2, 7\}, \{0, 3, 5\}, \{0, 3, 8\}, \{1, 2, 6\}, \{1, 2, 9\}, \{1, 3, 4\}, \{1, 3, 7\}, \{0, 4, 8\}, \{0, 5, 9\}, \{0, 6, 7\}, \{0, 6, 9\}, \{1, 4, 9\}, \{1, 5, 7\}, \{1, 5, 8\}, \{1, 6, 8\}, \{2, 4, 8\}, \{2, 5, 8\}, \{2, 5, 9\}, \{2, 6, 7\}, \{3, 4, 9\}, \{3, 5, 7\}, \{3, 6, 8\}, \{3, 6, 9\}, \{4, 5, 6\}, \{7, 8, 9\}\}$. Then $(V = \{\infty_0\} \cup \mathbb{Z}_{10}, B)$ is a maximum packing of $2K_{11}$ with leave $(4, 7)$ in which the neighborhood of ∞_0 is $C_2 \cup C_2 \cup C_3 \cup C_3$. Since the triples $\{0, 4, 8\}, \{1, 5, 8\} \in B$, apply Lemma 3.5 to (V, B) with $v = \infty_0$, $w = 8$, $a = 0$, $b = 1$, $c = 4$, and $e = 5$, to replace the cycles $(0, 1)$ and $(4, 5, 6)$ in the neighborhood of ∞_0 with the cycle $(0, 1, 5, 6, 4)$, so in the resulting maximum packing (V, B') the neighborhood of ∞_0 is $C_5 \cup C_2 \cup C_3$. Finally, since the triples $\{2, 7, 6\}, \{3, 9, 6\} \in B'$, apply Lemma 3.5 to (V, B') with $v = \infty_0$, $w = 6$, $a = 2$, $b = 3$, $c = 7$, and $e = 9$, to replace the cycles $(2, 3)$ and $(7, 8, 9)$ in the neighborhood of ∞_0 with the cycle $(2, 3, 9, 8, 7)$, so in the resulting maximum packing (V, B'') the neighborhood of ∞_0 is $C_5 \cup C_5$. □

3.4 Quadratic Leaves

Having proved Theorem 3.9, it can now be used along with the corresponding Colbourn and Rosa result (see [9]) to assist in proving the second main result of this chapter.

Theorem 3.10. *Let Q be a quadratic graph in λK_n. There exists a K_3-decomposition of $\lambda K_n - E(Q)$ if and only if*

1. *$\lambda(n-1)$ is even,*

2. *$|E(\lambda K_n)| - |E(Q)|$ is divisible by 3,*

3. *$(\lambda, n, Q) \notin \{(1, 7, C_3 \cup C_3), (1, 9, C_4 \cup C_5), (2, 6, C_3 \cup C_3), (2, 5, C_2 \cup C_3)\}$, and*

4. if $\lambda \neq 2$ then $n \neq 2$.

Proof. To see the necessity of Conditions $(1-4)$ consider the following. If $(\lambda, n, Q) \in \{(1,7,C_3 \cup C_3),(1,9,C_4 \cup C_5)\}$ then by the Colbourn and Rosa result (see Theorem 1.2), there is no K_3-decomposition of $K_n - E(Q)$. If $(\lambda, n, Q) \in \{(2,5,C_2 \cup C_3),(2,6,C_3 \cup C_3)\}$ and if there exists a K_3-decomposition $(\mathbb{Z}_n, B_1)$ of $2K_n - E(Q)$ then $(\mathbb{Z}_{n+1}, B_1 \cup \{\{n,a,b\} \mid \{a,b\} \in E(Q)\})$ is a K_3-decomposition of $2K_{n+1}$ in which the neighborhood of n is Q; but this contradicts Theorem 1.1. This proves the necessity of Condition (3). The necessity of Conditions (1) and (2) follows since each vertex in each triple has even degree and each triple contains 3 edges respectively. The necessity of Condition (4) is clear since λK_2 contains no copies of K_3.

To prove the sufficiency, the result is clear if $n \leq 2$, and the result follows from Theorem 1.2 if $\lambda = 1$, so assume that $n \geq 3$ and $\lambda \geq 2$. Let Q be a quadratic graph such that Conditions $(1-3)$ are satisfied.

Case 1: Suppose $\lambda = 2$.

If $(n, Q) \in \{(5, C_2),(6, C_3)\}$ then the result follows in each case from Lemma 3.2 by taking a maximum packing of $2K_n$, then removing any one triple if $n = 6$. By Condition 2, $|E(Q)| \equiv n - \epsilon \pmod 3$ where $\epsilon = 1$ if $n \equiv 1 \pmod 3$ and $\epsilon = 0$ otherwise. Form the 2-regular graph Q' on $n-\epsilon$ vertices by adding $\frac{n-|E(Q)|-\epsilon}{3}$ 3-cycles to Q (by Condition (2) this is an integral number of 3-cycles). Let $B = \{\{x,y,z\} \mid (x,y,z) \in Q' \setminus Q\}$. It can be assumed that $(n+1, Q') \notin \{(7, C_3 \cup C_3),(6, C_2 \cup C_3)\}$ since $(n+1, Q') \in \{(7, C_3 \cup C_3),(6, C_2 \cup C_3)\}$ only if $(n, Q) \in \{(5, C_2),(6, C_3),(5, C_2 \cup C_3),(6, C_3 \cup C_3)\}$, where the first two are handled at the beginning of this case and the last two are prohibited by Condition 3. So by either Theorem 1.1 or 3.9 there exists a maximum packing $(\mathbb{Z}_{n+1}, B')$ of $2K_{n+1}$ in which the neighborhood of the vertex n is Q'. Then $(\mathbb{Z}_n, (B' \setminus \{b \in B' \mid n \in b\}) \cup B)$ is the required decomposition.

Case 2: Suppose $\lambda > 2$ and $n \equiv 0$ or $1 \pmod 3$. First suppose that $(n, Q) \neq (6, C_3 \cup C_3)$. By Condition 2, $|E(Q)| \equiv 0 \pmod 3$ regardless of the value of λ. By the result of Case 1, let $(\mathbb{Z}_n, B_0)$ be a K_3-decomposition of $2K_n - E(Q)$. Let $(\mathbb{Z}_n, B_1)$ be a K_3-decomposition of

$(\lambda-2)K_n$; this exists by Lemma 3.2 since $n \equiv 0$ or 1 (mod 3) and since if n is even then $\lambda-2$ is even by Condition (1). Then $(\mathbb{Z}_n, B_0 \cup B_1)$ is the required decomposition. Using Lemma 3.2, if $(n, Q) = (6, C_3 \cup C_3)$ then let $(\mathbb{Z}_6, B_1)$ and $(\mathbb{Z}_6, B_2)$ be K_3-decompositions of $2K_6$ such that $\{0,1,2\} \in B_1$ and $\{3,4,5\} \in B_2$, and let $(\mathbb{Z}_6, B_3)$ be a K_3-decomposition of $(\lambda-4)K_6$. Then $(\mathbb{Z}_6, B_1 \cup B_2 \cup B_3 \setminus \{\{0,1,2\},\{3,4,5\}\})$ is the required decomposition.

Case 3: Suppose $\lambda > 2$ and $n \equiv 2$ (mod 3). Let $V(Q) \subset \mathbb{Z}_n$. By Condition (4), $n \neq 2$. First suppose that $n \neq 5$. Let $\delta \equiv \lambda$ (mod 3) with $\delta \in \{2,3,4\}$ if $n \equiv 5$ (mod 6) and $\delta \in \{2,4,6\}$ if $n \equiv 2$ (mod 6). By Lemma 3.2 and Conditions (1) and (2) there exists a K_3-decomposition $(\mathbb{Z}_n, B_0)$ of $(\lambda-\delta)K_n$.

If $\delta = 2$ then by Condition (2) $|E(Q)| \equiv 2$ (mod 3), so by Case 1 let $(\mathbb{Z}_n, B_1)$ be a K_3-decomposition of $2K_n - E(Q)$.

If $\delta = 4$ then by Condition (2) $|E(Q)| \equiv 1$ (mod 3) so $|E(Q)| \leq n-1$, so say $0 \notin V(Q)$ and that $(1,\ldots,x)$ is a cycle in Q. Form Q' from Q by replacing the cycle $(1,\ldots,x)$ in Q with the cycle $(0,1,\ldots,x)$. By Case 1 there exists a K_3-decomposition $(\mathbb{Z}_n, B_1')$ of $2K_n - E(Q')$. By Lemma 3.2 let $(\mathbb{Z}_n, B_1'')$ be a maximum packing of $2K_n$ with leave the 2-cycle $(1,x)$. Let $B_1 = B_1' \cup B_1'' \cup \{0,1,x\}$.

If $\delta \in \{3,6\}$ then by Condition (2) $|E(Q)| \equiv 0$ (mod 3), so say $0,1 \notin V(Q)$. Form Q' from Q by adding the 2-cycle $(0,1)$. By Case 1 let $(\mathbb{Z}_n, B_1')$ be a K_3-decomposition of $2K_n - E(Q')$. By Lemma 3.2 let $(\mathbb{Z}_n, B_1'')$ be a maximum packing of $(\delta-2)K_n$ with leave the 4-cycle $(2,0,3,1)$. Let $B_1 = B_1' \cup B_1'' \cup \{\{0,1,2\},\{0,1,3\}\}$.

Then for each $\delta \in \{2,3,4,6\}$, $(\mathbb{Z}_n, B_0 \cup B_1)$ is the required decomposition.

Finally, suppose that $n = 5$. Based on the above argument, it suffices to find packings of λK_5 with leave Q for $(\lambda, Q) \in \{(2, C_5), (3, C_3), (4, C_4), (4, C_2 \cup C_2), (5, C_2 \cup C_3)\}$. $(\lambda, Q) = (2, C_5)$ was handled in Case 1. When $(\lambda, Q) = (3, C_3)$, by Lemma 3.2 let $(\mathbb{Z}_5, B)$ be a K_3-decomposition of $3K_5$ and let $b \in B$; then $(\mathbb{Z}_5, B \setminus b)$ is the required decomposition. When $(\lambda, Q) = (4, C_4)$ the result follows from Lemma 3.2 by taking a maximum packing of $4K_5$. When $(\lambda, Q) = (4, C_2 \cup C_2)$, by Lemma 3.2 let $(\mathbb{Z}_5, B_1)$ and $(\mathbb{Z}_5, B_2)$ be maximum packings of

$2K_5$ with leaves $(0,1)$ and $(2,3)$ respectively; then $(\mathbb{Z}_5, B_1 \cup B_2)$ is the required decomposition. Finally, when $(\lambda, Q) = (5, C_2 \cup C_3)$, by Lemma 3.2 let $(\mathbb{Z}_5, B_1)$ be a K_3-decomposition of $3K_5$ that contains the triple $\{0,1,2\}$, and let $(\mathbb{Z}_5, B_2)$ be a maximum packing of $2K_5$ with leave the 2-cycle $(3,4)$; then $(\mathbb{Z}_5, B_1 \cup B_2 \setminus \{0,1,2\})$ is the required decomposition. $\square$

Chapter 4

Neighborhoods in Maximum Packings of $2K_n$- A Second Version

4.1 Introduction

Having completed the solution to the neighborhood graph problem in the last chapter by solving the case where $n \equiv 2 \pmod 3$, this chapter focuses on finding a unified proof for the neighborhood graph problem. While the proof technique used in the last chapter for $n \equiv 2 \pmod 3$ is similar in principle to the proof technique used in the corresponding Colbourn and Rosa result, it does not seem that the techniques used in either proof can be used to readily obtain the other result, even if one "allows" extreme cases (such as the case when each cycle in the neighborhood has length two) to be handled using alternate methods. In this chapter, a new, simpler, and unified proof that obtains both results is provided (see Theorem 4.5). However, this new proof relies heavily on a major result, namely a recent and quite powerful result due to Bryant, Horsley, and Pettersson (see Theorem 4.3). Section 4.2 will begin with some well-known lemmas that are useful in handling extreme cases of Theorem 4.5. The theorem of Bryant, Horsley, and Pettersson will then be given and used to establish a lemma that will be used in several cases of the proof of the main theorem. Finally, Section 4.3 contains the new proof of the main theorem.

4.2 Preliminary Results

To begin this section, two well-known results are given, one being on idempotent quasigroups and the other on maximum partial triple systems. These lemmas will be used to handle extreme cases of the main theorem (specifically the cases where $n \equiv 1$ or $5 \pmod 6$) and Q contains only 2-cycles).

Lemma 4.1. [27] *There exists an idempotent quasigroup of order n for all $n \neq 2$.*

The second lemma is more extensive than what appears below; however, what appears below is sufficient for this chapter.

Lemma 4.2. [14] *The leave of a maximum partial triple system of λK_n is*

1. *$\varnothing$ if $\lambda = 2$ and $n \equiv 0, 1 \pmod 3$,*
2. *a 2-cycle if $\lambda = 2$ and $n \equiv 2 \pmod 3$,*
3. *a $K_{1,3}$ and $\frac{n-4}{2}$ independent edges if $\lambda = 1$ and $n \equiv 4 \pmod 6$, and*
4. *a 1-factor if $\lambda = 1$ and $n \equiv 2 \pmod 6$.*

The powerful cycle-decomposition theorem of Bryant, Horsley, and Petterson from [5] appears next.

Theorem 4.3. *[5]*

1. *Let n be odd. There exists a decomposition of K_n into cycles of length $m_1, \ldots, m_t$ if and only if*
   (a) *$3 \leq m_i \leq n$ for $1 \leq i \leq t$ and*
   (b) *$\sum_{i=1}^{t} m_i = \binom{n}{2}$.*
2. *Let n be even. There exists a decomposition of K_n into cycles of length $m_1, \ldots, m_t$ and a 1-factor F if and only if*
   (a) *$3 \leq m_i \leq n$ for $1 \leq i \leq t$ and*
   (b) *$\sum_{i=1}^{t} m_i = \binom{n}{2} - \frac{n}{2}$.*

This result provides the backbone of the proof technique used in this chapter, establishing that K_n minus the edges of a certain set of cycles and possibly a 1-factor can be decomposed into triples.

The full power of this result is not needed for the main theorem, since at most three cycle lengths are chosen for any particular case. However, while older and more basic results can be used in many of the cases, it does not seem like older results are sufficient to handle all cases in the proof of the theorem (for instance, the case in which K_n needs to be decomposed into a 1-factor, a Hamilton cycle, a near Hamilton cycle, and triples.)

A lemma that will be used in multiple cases in the proof of Theorem 4.5 is now given. The lemma and the proof of it will look quite similar to the proofs that appear in the cases of the proof of the main theorem; this lemma is stated here however since it is used in several cases of the main theorem.

Lemma 4.4. *Let $n \equiv 0$ (mod 3) and let Q be a 2-regular graph on n vertices. Then for any integer $2 \leq k \leq n-1$, there exists a decomposition of $2K_n$ into triples and k 2-factors, one of which is Q.*

Proof. Let $\epsilon \in \{0,1\}$ with $\epsilon \equiv n \pmod 2$. Let $Q = \{c_0, \dots, c_{q-1}\}$, where for each $i \in \mathbb{Z}_q$, the cycle $c_i = (c_{i,1}, \dots, c_{i,l_i})$ has length l_i. Since $n \equiv 0 \pmod 3$, the number of edges in any Hamilton cycle of K_n and any 1-factor of K_n is a multiple of 3. Further, since $|E(K_n)| \equiv 0 \pmod 3$, by Theorem 4.3, K_n can be decomposed into j Hamilton cycles for $0 \leq j \leq \frac{n-2+\epsilon}{2}$, $1-\epsilon$ 1-factors, and triples. So let $(\mathbb{Z}_n, B_1)$ be a K_3-decomposition of $K_n - E(G_1)$, where G_1 consists of the $h = \min\{k-1, \frac{n-2+\epsilon}{2}\}$ Hamilton cycles $H_1, \dots, H_h$, along with the 1-factor F_0 if $\epsilon = 0$. In particular, let $H_1 = (c_{0,1}, \dots, c_{0,l_0}, c_{1,1}, \dots, c_{1,l_1}, \dots, c_{q-1,1}, \dots, c_{q-1,l_{q-1}})$. Let $(\mathbb{Z}_n, B_2)$ be a K_3-decomposition of $K_n - E(G_2)$ where G_2 contains the $k-h-1+\epsilon$ Hamilton cycles $H_{h+2-\epsilon}, \dots, H_k$, along with the 1-factor F_1 if $\epsilon = 0$. Note that if $\epsilon = 1$, then $k-h-1+\epsilon = k-h \geq k-(k-1) = 1$ so G_2 contains a Hamilton cycle in this case. If $\epsilon = 1$ then let $H_{h+1} = (c_{0,1}, c_{0,l_0}, c_{1,1}, c_{1,l_1}, \dots, c_{q-1,1}, c_{q-1,l_{q-1}}, v_1, \dots, v_{n-2q})$ where $v_1, \dots, v_{n-2q}$ are arbitrarily named. If $\epsilon = 0$ then name the vertices so that q of the edges in F_1 are in $\{\{c_{i,1}, c_{i,l_i}\} \mid i \in \mathbb{Z}_q\}$ and let $H_{h+1} = F_0 \cup F_1$.

Let H_1' be the 2-factor induced by $(E(H_1) \cup \{\{c_{i,1}, c_{i,l_i}\} \mid i \in \mathbb{Z}_q\}) \setminus \{\{c_{i,l_i}, c_{i+1,1}\} \mid i \in \mathbb{Z}_q\}$ reducing the sum in the subscript modulo q. Then $H_1' \equiv Q$. The graph H_{h+1}' induced by

$E(H_1) \cup E(H_{h+1}) - E(H_1')$ is a 2-factor. Finally, for $i \in \{1, 2, \ldots, k\} \setminus \{1, h+1\}$, let $H_i' = H_i$. Then $(\mathbb{Z}_n, B_1 \cup B_2 \cup \cup_{i \in \mathbb{Z}_k} H_i')$ is the required decomposition. $\square$

4.3 Main Result

The main theorem is now stated and proved. The theorem was previously proved by Colbourn and Rosa in [9] when $n \equiv 0$ or 1 (mod 3) and in Chapter 3 when $n \equiv 2$ (mod 3).

Theorem 4.5. *[7, 9] Suppose $n \neq 2$. Let Q be a 2-regular multigraph on $n - 1 - \alpha$ vertices with $\alpha \in \{0, 1\}$. Then there exists a maximum packing of $2K_n$ (possibly the leave is empty) such that the neighborhood graph of some vertex is Q if and only if*

1. *$\alpha = 0$ if $n \equiv 0$ or 1 (mod 3) and*
2. *$(n, Q) \notin \{(5, C_2 \cup C_2), (6, C_2 \cup C_3), (7, C_3 \cup C_3)\}$.*

Proof. If $n \equiv 0$ or 1 (mod 3), then a maximum packing of $2K_n$ is a 2-fold triple system, so the neighborhood graph of every vertex will be a 2-regular graph on $n - 1$ vertices so $\alpha = 0$ in these cases, which shows the first condition is necessary.

If $(n, Q) = (5, C_2 \cup C_2), (6, C_2 \cup C_3)$, or $(7, C_3 \cup C_3)$ and there exists a maximum packing of $2K_n$ such that the leave of some vertex v is Q, then deleting the triples containing v leaves the graph $\overline{K_2} \bigvee_2 \overline{K_2}$, $\overline{K_2} \bigvee_2 K_3$, or $K_3 \bigvee_2 K_3$ respectively. So each graph can be thought of as a graph with two parts with 2 edges joining each pair of vertices in different parts. In each case, any triple that contains an edge with its incident vertices in different parts (a mixed edge) contains 2 such edges and 1 edge whose incident vertices lie in the same part (a pure edge). But in each case, there are more than twice as many mixed edges as pure edges remaining. Hence these cases are not possible.

To prove the sufficiency, two extreme cases (Case 1) will be followed by 9 main cases. While this is a large number of cases, most follow from Lemma 4.4 or from similar ideas. Let Q be a 2-regular multigraph on $n - 1 - \alpha$ vertices (with the size of Q specified in each case) such that $(n, Q) \notin \{(5, C_2 \cup C_2), (6, C_2 \cup C_3), (7, C_3 \cup C_3)\}$.

Case 1: Suppose $n \equiv 1$ or $5 \pmod 6$ and that Q consists entirely of 2-cycles (so since n is odd then $\alpha = 0$). By assumption $(n, Q) \neq (5, C_2 \cup C_2)$, and the case where $n = 1$ is trivial. So it can be assumed that $n \geq 7$; let $n = 6l + 1 + \epsilon$ with $l \geq 1$ and $\epsilon \in \{0, 4\}$. Let $V = \{\infty_1\} \cup (\mathbb{Z}_{3l+\frac{\epsilon}{2}} \times \mathbb{Z}_2)$. By Lemma 4.1, let $(\mathbb{Z}_{3l+\frac{\epsilon}{2}}, \circ)$ be an idempotent quasigroup ($l \geq 1$, so $3l + \frac{\epsilon}{2} \geq 3$, so the quasigroup exists). By Lemma 4.2, there exists a maximum packing $(\mathbb{Z}_{3l+\frac{\epsilon}{2}} \times \{1\}, B_1)$ of $2K_{3l+\frac{\epsilon}{2}}$ with leave c where c is a 2-cycle if $\epsilon = 4$ and $c = \varnothing$ otherwise. Then $(\{\infty_1\} \cup (\mathbb{Z}_{3l+\frac{\epsilon}{2}} \times \mathbb{Z}_2), B_1 \cup \{\{(a,0),(b,0),(a \circ b,1)\}, \{(a,0),(b,0),(b \circ a,1)\} \mid 0 \leq a < b \leq 3l - 1 + \frac{\epsilon}{2}\} \cup \{\{\infty_1,(a,0),(a,1)\}, \{\infty_1,(a,0),(a,1)\} \mid a \in \mathbb{Z}_{3l+\frac{\epsilon}{2}}\})$ is the required decomposition with leave c.

Case 2: Suppose $n \equiv 2 \pmod 3$ with $\alpha = 1$. A maximum packing is constructed on the vertex set $\mathbb{Z}_{n-2} \cup \{\infty_1, \infty_2\}$ where the neighborhood of ∞_1 is $Q = \{c_0, \ldots, c_{q-1}\}$, the q-cycles being defined on the vertex set $\mathbb{Z}_{n-2}$; so the leave of the maximum packing will be (∞_1, ∞_2).

In this case, $w = n - 2 \equiv 0 \pmod 3$ and $2 \leq w - 1$ (since $n \equiv 2 \pmod 3$ and $n \neq 2$). So by Lemma 4.4, there exists a K_3-decomposition $(\mathbb{Z}_{n-2}, B)$ of $2K_{n-2} - (E(H_1') \cup E(H_2'))$ where H_1' and H_2' are 2-regular graphs and $H_1' \equiv Q$.

Then $(\mathbb{Z}_{n-2} \cup \{\infty_1, \infty_2\}, B \cup \{\{\infty_i, a_i, b_i\} \mid \{a_i, b_i\} \in E(H_i'), 1 \leq i \leq 2\})$ is the required maximum packing.

Case 3: Suppose $n \equiv 2 \pmod 3$, $\alpha = 0$, and that Q has a cycle of length at least $5 - \epsilon$ where $\epsilon \in \{0, 1\}$ with $\epsilon \equiv n \pmod 2$.

A maximum packing of $2K_n$ with leave a 2-cycle will be constructed on the vertex set $\mathbb{Z}_{n-2} \cup \{\infty_1, \infty_2\}$ such that the neighborhood of the vertex ∞_1 is Q.

Let $Q = \{c_0, ..., c_{q-1}\}$ where for each $i \in \mathbb{Z}_q$, the cycle $c_i = (c_{i,1}, \ldots, c_{i,l_i})$ has length l_i, $c_{j,k} \in \mathbb{Z}_{n-2}$ unless $(j,k) = (0,3)$, $c_{0,3} = \infty_2$, and $l_0 \geq 5 - \epsilon$. Since $n - 2 \equiv 0$ (mod 3), the number of edges in any Hamilton cycle of K_{n-2} and any 1-factor of K_{n-2} is a multiple of 3. Further, since $|E(K_{n-2})| \equiv 0 \pmod 3$, by Theorem 4.3, K_{n-2} can be decomposed into 0 or 1 Hamilton cycles, $1 - \epsilon$ 1-factors, and triples. So let $(\mathbb{Z}_{n-2}, B_1)$

be a K_3-decomposition of $K_{n-2} - E(G_1)$, where G_1 consists of the Hamilton cycle $H_1 = (c_{0,1}, c_{0,2}, c_{0,4}, \ldots, c_{0,l_0}, c_{1,1}, \ldots, c_{1,l_1}, \ldots, c_{q-1,1}, \ldots, c_{q-1,l_{q-1}})$ ($c_{0,3} = \infty_2 \notin \mathbb{Z}_{n-2}$), along with the 1-factor F_0 if $\epsilon = 0$. If $\epsilon = 1$ then let $(\mathbb{Z}_{n-2}, B_2)$ be a K_3-decomposition of $K_{n-2} - E(H_2)$ where H_2 is the Hamilton cycle $H_2 = (c_{0,2}, c_{0,4}, c_{0,1}, c_{0,l_0}, c_{1,1}, c_{1,l_1}, \ldots, c_{q-1,1}, c_{q-1,l_{q-1}}, v_1, \ldots, v_{(n-2)-2-2q})$ with $v_1, \ldots, v_{(n-2)-2-2q}$ arbitrarily named if $l_0 \geq 5$; if $l_0 = 4$, drop the extra occurrence of c_{0,l_0} after $c_{0,1}$ and have one more arbitrarily named vertex at the end of H_2. If $\epsilon = 0$ then let $(\mathbb{Z}_{n-2}, B_2)$ be a K_3-decomposition of $K_{n-2} - E(F_1)$ where F_1 is a 1-factor named so that $\{c_{0,2}, c_{0,4}\}$ is an edge and q of the remaining edges in F_1 are in $\{\{c_{i,1}, c_{i,l_i}\} \mid i \in \mathbb{Z}_q\}$. (Note that $l_0 \geq 5$ if $\epsilon = 0$ so $c_{0,l_0} \neq c_{0,4}$.) If $\epsilon = 0$, let $H_2 = F_0 \cup F_1$. Finally, observe that the edge $\{c_{0,2}, c_{0,4}\}$ appears in both $E(H_1)$ and $E(H_2)$.

Let H_1' be the graph induced by $(E(H_1) \cup \{\{c_{i,1}, c_{i,l_i}\} \mid i \in \mathbb{Z}_q\}) \setminus \{\{c_{i,l_i}, c_{i+1,1}\} \mid i \in \mathbb{Z}_q\}$ reducing the sum in the subscript modulo q. Let H_2' be the graph induced by $(E(H_1) \cup E(H_2)) \setminus E(H_1')$. Then H_1' and H_2' are each 2-regular spanning subgraphs of K_{n-2}, and the set of cycles formed by the components in H_1' contains the cycles $c_1, ..., c_{q-1}$ and the cycle c_0' where c_0' is formed from c_0 by deleting the edges $\{c_{0,2}, c_{0,3}\}$ and $\{c_{0,3}, c_{0,4}\}$ and adding the edge $\{c_{0,2}, c_{0,4}\}$. Note that the edge $\{c_{0,2}, c_{0,4}\}$ appears in both $E(H_1')$ and $E(H_2')$.

Then $(\mathbb{Z}_{n-2} \cup \{\infty_1, \infty_2\}, (B_1 \cup B_2 \cup \{\{\infty_i, a_i, b_i\} \mid \{a_i, b_i\} \in E(H_i'), 1 \leq i \leq 2\} \setminus \{\{\infty_i, c_{0,2}, c_{0,4}\} \mid i \in \{1,2\}\}) \cup \{\{\infty_1, \infty_2, c_{0,i}\} \mid i \in \{2,4\}\})$ is the required maximum packing (with leave $(c_{0,2}, c_{0,4})$).

Case 4: Suppose that $n \equiv 5 \pmod 6$, $\alpha = 0$, and that Q contains a 3-cycle.

A maximum packing will be constructed on the vertex set $\mathbb{Z}_{n-4} \cup \{\infty_j \mid 1 \leq j \leq 4\}$ where the neighborhood of ∞_1 is $Q = \{c_0, \ldots, c_{q-1}\}$, where $c_0 = (\infty_2, \infty_3, \infty_4)$, and the $q-1$ other cycles are defined on the vertex set $\mathbb{Z}_{n-4}$ with l_1 being odd (since $n-1-\alpha$ is even, and Q contains a 3-cycle, Q must contain some other cycle of odd length). The leave of the maximum packing will be $(\infty_2, c_{1,2})$, where $c_{1,2}$ is defined below.

For each $i \in \mathbb{Z}_q \setminus \{0\}$, let $c_i = (c_{i,1}, \ldots, c_{i,l_i})$ where l_i is the length of c_i. In this case, $n-4 \equiv 1 \pmod 6$ and thus $\binom{(n-4)}{2} - 3(n-4)$ and $\binom{(n-4)}{2} - (n-4-1)$ are both divisible by 3. Further, Q cannot contain a 3-cycle in this case if $n = 5$, so $n - 4 \geq 7$, so both quantities are also nonnegative. So by Theorem 4.3, let $(\mathbb{Z}_{n-4}, B_1)$ be a K_3-decomposition of $K_{n-4} - (E(H_1) \cup E(H_3) \cup E(H_4))$ and let $(\mathbb{Z}_{n-4}, B_2)$ be a K_3-decomposition of $K_{n-4} - E(H_2)$ where H_1, H_3, and H_4 are Hamilton cycles and H_2 is a near-Hamilton cycle with H_1 and H_2 named as follows: Let $H_1 = (c_{1,1}, \ldots, c_{1,l_1}, c_{2,1}, \ldots, c_{2,l_2}, \ldots, c_{q-1,1}, \ldots, c_{q-1,l_{q-1}})$. Let H_2 be defined by $H_2 = (c_{1,1}, c_{1,l_1}, c_{2,1}, c_{2,l_2}, \ldots, c_{q-1,1}, c_{q-1,l_{q-1}}, v_1, \ldots, v_{n-3-2q})$ where $v_1, \ldots, v_{n-3-2q}$ exclude $c_{1,2}$ and are otherwise arbitrarily named (note that $c_{1,2}$ is omitted from H_2 altogether since l_1 is odd and hence $c_{1,2} \neq c_{1,l_l}$).

Let H_1' be the graph induced by $(E(H_1) \cup \{\{c_{i,1}, c_{i,l_i}\} \mid i \in \mathbb{Z}_q \setminus \{0\}\}) \setminus (\{\{c_{i,l_i}, c_{i+1,1}\} \mid i \in \mathbb{Z}_q \setminus \{0, q-1\}\} \cup \{\{c_{q-1,l_{q-1}}, c_{1,1}\}\})$. Let H_2' be the graph induced by $(E(H_1) \cup E(H_2)) \setminus E(H_1')$. Then H_1' and H_2' are 2-regular spanning subgraphs of K_{n-4} and K_{n-5} respectively, with the set of cycles formed by the components in H_1' being $Q \setminus c_0$. Let $H_3' = H_3$ and $H_4' = H_4$.

Let $(\{\infty_j \mid 1 \leq j \leq 4\}, B_3)$ be a K_3-decomposition of $2K_4$ (where the neighborhood of ∞_1 is the 3-cycle $(\infty_2, \infty_3, \infty_4)$).

Then $(\mathbb{Z}_{n-4} \cup \{\infty_j \mid 1 \leq j \leq 4\}, B_1 \cup B_2 \cup B_3 \cup \{\{\infty_i, a_i, b_i\} \mid \{a_i, b_i\} \in E(H_i')\})$ is the required packing.

Case 5: Suppose that $n \equiv 2 \pmod 6$, $\alpha = 0$ and that Q contains a 3-cycle.

First suppose $n = 8$. Let $B = \{\{\infty_1, 0, 1\}, \{\infty_1, 0, 1\}, \{\infty_1, 2, 3\}, \{\infty_1, 2, 3\}, \{\infty_1, 4, 5\}, \{\infty_1, 4, 6\}, \{\infty_1, 5, 6\}, \{0, 2, 4\}, \{0, 2, 5\}, \{0, 3, 4\}, \{0, 3, 5\}, \{1, 2, 4\}, \{1, 2, 6\}, \{1, 3, 5\}, \{1, 3, 6\}, \{1, 4, 5\}, \{2, 5, 6\}, \{3, 4, 6\}\}$. Then $(\mathbb{Z}_7 \cup \{\infty_1\}, B)$ is a maximum packing of $2K_8$ with leave $(0, 6)$ such that the neighborhood of ∞_1 is $C_2 \cup C_2 \cup C_3$. Finally, $(\mathbb{Z}_7 \cup \{\infty_1\}, (B \setminus \{\{\infty_1, 0, 1\}, \{\infty_1, 2, 3\}, \{0, 2, 5\}, \{1, 3, 5\}\}) \cup \{\{\infty_1, 0, 2\}, \{\infty_1, 1, 3\}, \{0, 1, 5\}, \{2, 3, 5\}\})$ is a maximum packing of $2K_8$ such that the neighborhood of ∞_1 is $C_4 \cup C_3$.

For $n > 8$, a maximum packing is constructed on the vertex set $\mathbb{Z}_{n-4} \cup \{\infty_j \mid 1 \leq j \leq 4\}$ where the neighborhood of ∞_1 is $Q = \{c_0, \ldots, c_{q-1}\}$, where $c_0 = (\infty_2, \infty_3, \infty_4)$, and the $q-1$ other-cycles are defined on the vertex set $\mathbb{Z}_{n-4}$. The maximum packing will be constructed so that the leave will be (∞_3, a), where a is defined below.

For each $i \in (\mathbb{Z}_q \setminus \{0\})$, let $c_i = (c_{i,1}, \ldots, c_{i,l_i})$ where l_i is the length of c_i. In this case, $n-4 \equiv 4 \pmod 6$ and ≥ 10 and thus $\binom{(n-4)}{2} - (n-4) - \frac{n-4}{2} - (n-4-1)$ and $\binom{(n-4)}{2} - (n-4) - \frac{n-4}{2}$ are both divisible by 3 and nonnegative. So by Theorem 4.3, let $(\mathbb{Z}_{n-4}, B_1)$ be a K_3-decomposition of $K_{n-4} - (E(H_1) \cup E(H_3) \cup E(F_1))$ and let $(\mathbb{Z}_{n-4}, B_2)$ be a K_3-decomposition of $K_{n-4} - (E(H_2) \cup E(F_2))$ where H_1 and H_2 are Hamilton cycles, H_3 is a near-Hamilton cycle with arbitrarily named vertex $a \in \mathbb{Z}_{n-4}$ omitted, F_1 and F_2 are 1-factors, and H_1 and H_2 are named as follows: Let $H_1 = (c_{1,1}, \ldots, c_{1,l_1}, c_{2,1}, \ldots, c_{2,l_2}, \ldots, c_{q-1,1}, \ldots, c_{q-1,l_{q-1}})$. Let $H_2 = (c_{1,1}, c_{1,l_1}, c_{2,1}, c_{2,l_2}, \ldots, c_{q-1,1}, c_{q-1,l_{q-1}}, v_1, \ldots, v_{n+2-2q})$ where $v_1, \ldots, v_{n+2-2q}$ are arbitrarily named.

Let H_1' be the graph induced by $(E(H_1) \cup \{\{c_{i,1}, c_{i,l_i}\} \mid i \in \mathbb{Z}_q \setminus \{0\}\}) \setminus (\{\{c_{i,l_i}, c_{i+1,1}\} \mid i \in \mathbb{Z}_q \setminus \{0, q-1\}\} \cup \{\{c_{q-1,l_{q-1}}, c_{1,1}\}\})$. Let H_2 be the graph induced by $(E(H_1) \cup E(H_2)) \setminus E(H_1')$. Then H_1' and H_2' are 2-regular spanning subgraphs of K_{n-4}, with the set of cycles formed by the components in H_1' being $Q \setminus c_0$. Let $H_3' = H_3$ and H_4' be the graph induced by $E(F_1) \cup E(F_2)$.

Let $(\{\infty_j \mid 1 \leq j \leq 4\}, B_3)$ be a K_3-decomposition of $2K_4$ (where the neighborhood of ∞_1 is the 3-cycle $(\infty_2, \infty_3, \infty_4)$).

Then $(\mathbb{Z}_{n-4} \cup \{\infty_j \mid 1 \leq j \leq 4\}, B_1 \cup B_2 \cup B_3 \cup \{\{\infty_i, a_i, b_i\} \mid \{a_i, b_i\} \in E(H_i')\})$ is the required decomposition.

Case 6: Suppose $n \equiv 0 \pmod 3$, $\alpha = 0$, and that Q has a 2-cycle.

A maximum packing is constructed on the vertex set $\mathbb{Z}_{n-3} \cup \{\{\infty_j\} \mid 1 \leq j \leq 3\}$ where the neighborhood of ∞_1 is $Q = \{c_0, \ldots, c_{q-1}\}$, where $c_0 = (\infty_2, \infty_3)$, and the $q-1$ other-cycles are defined on the vertex set $\mathbb{Z}_{n-3}$.

Note that $n \neq 6$, since otherwise $Q = C_2 \cup C_3$ and $(n, Q) \neq (6, C_2 \cup C_3)$ by assumption. The case $n \equiv 3$ is trivial. Otherwise $w = n - 3 \equiv 0 \pmod 3$ and $3 \leq w - 1$, so by Lemma 4.4, there exists a K_3-decomposition $(\mathbb{Z}_{n-3}, B)$ of $2K_{n-3} - (E(H_1') \cup E(H_2') \cup E(H_3'))$ where H_1', H_2', and H_3' are 2-regular graphs and $H_1' \equiv Q \setminus c_0$.

Let $(\{\{\infty_j\} \mid 1 \le j \le 3\}, B_3)$ be a K_3-decomposition of $2K_3$ (where the neighborhood of ∞_1 is the 2-cycle (∞_2, ∞_3)).

Then $(\mathbb{Z}_{n-3} \cup \{\{\infty_j\} \mid 1 \le j \le 3\}, B_1 \cup B_2 \cup B_3 \cup \{\{\infty_i, a_i, b_i\} \mid \{a_i, b_i\} \in E(H_i')\})$ is the required decomposition.

Case 7: Suppose $n \equiv 0 \pmod 6$, $\alpha = 0$, and that Q has no cycles of length 2 (and hence one of length at least 4).

First, suppose $n = 6$. Define $B = \{\{\infty_1, 5, 0\}, \{\infty_1, 5, 1\}, \{\infty_1, 0, 2\}, \{\infty_1, 1, 3\}, \{\infty_1, 2, 3\}, \{5, 0, 3\}, \{5, 1, 2\}, \{5, 2, 3\}, \{0, 1, 2\}, \{0, 1, 3\}\}$. Then $(\mathbb{Z}_5 \cup \{\infty_1\}, B)$ is a maximum packing of $2K_6$ such that the neighborhood of ∞_1 is C_5.

Otherwise for $n \ge 12$, a maximum packing is constructed on the vertex set $\mathbb{Z}_{n-2} \cup \{\{\infty_j\} \mid 1 \le j \le 2\}$ where the neighborhood of ∞_1 is $Q = \{c_0, \ldots, c_{q-1}\}$.

For each $i \in \mathbb{Z}_q$, let $c_i = (c_{i,1}, \ldots, c_{i,l_i})$ where l_i is the length of c_i, $c_{j,k} \in \mathbb{Z}_{n-2}$ for all $(j,k) \ne (0,2)$, and $c_{0,2} = \infty_2$. In this case, $n-2 \equiv 4 \pmod 6$ and thus $\binom{(n-2)}{2} - \frac{n-2}{2} - (n-2-3)$ and $\binom{(n-2)}{2} - \frac{n-2}{2} - 1$ are both divisible by 3, and since $n-2 \ge 10$, both quantities are nonnegative. So by Theorem 4.3, let $(\mathbb{Z}_{n-2}, B_1)$ be a K_3-decomposition of $K_{n-2} - (E(H_1) \cup E(F_1))$ and by Lemma 4.2, let $(\mathbb{Z}_{n-2}, B_2)$ be a K_3-decomposition of $K_{n-2} - E(H_2)$ where H_1 is a $n-5$ cycle, H_2 consists of a $K_{1,3}$ and $\frac{(n-2)-4}{2}$ independent edges, F_1 is a 1-factor, and H_1 and H_2 are named as follows: If $l_0 = 4$, let $H_1 = (c_{1,1}, \ldots, c_{1,l_1}, c_{2,1}, \ldots, c_{2,l_2}, \ldots, c_{q-1,1}, \ldots, c_{q-1,l_{q-1}})$ (so that $c_{0,1}, c_{0,3}$, and $c_{0,4}$ are omitted) and if $l_0 \ge 5$, let $H_1 = (c_{0,4}, c_{0,5}, \ldots, c_{0,l_0}, c_{1,1}, \ldots, c_{1,l_1}, c_{2,1}, \ldots, c_{2,l_2}, \ldots, c_{q-1,1}, \ldots, c_{q-1,l_{q-1}-1})$ (so that $c_{0,1}, c_{0,3}$, and $c_{q-1,l_{q-1}}$ are omitted). If $l_0 = 4$, let H_2 be defined to contain the edges $\{c_{i,1}, c_{i,l_i}\}$ for each $i \in \mathbb{Z}_q$ as well as the edges $\{c_{0,4}, c_{0,3}\}$ and $\{c_{0,4}, c_{1,2}\}$ (note that $c_{1,2} \ne c_{1,l_1}$ since all cycles have length greater than 2 and note that $c_{0,4}$ is the vertex of degree 3 in H_2) and finally $\frac{(n-2)-4-2(q-1)}{2}$ arbitrarily named edges. If $l_0 \ge 5$, let H_2 be defined to contain the edges $\{c_{i,1}, c_{i,l_i}\}$ for each $i \in \mathbb{Z}_q$ as well as the edges $\{c_{0,3}, c_{0,4}\}, \{c_{q-1,l_{q-1}}, c_{q-1,l_{q-1}-1}\}$, and $\{c_{q-1,l_{q-1}}, c_{1,2}\}$ (note that $c_{1,2} \ne c_{1,l_1}$ since all cycles have length greater than 2, $c_{0,4} \ne c_{0,l_0}$ since $l_0 \ge 5$, and $c_{q-1,l_{q-1}}$ is the vertex of degree 3 in H_2) and finally $\frac{(n-2)-4-2q}{2}$ arbitrarily named edges.

Note that in each case $\cup_{i=1}^{q-1} E(c_i) \cup E(c'_0) \subset E(H_1) \cup E(H_2)$ where c'_0 is formed from c_0 by removing the edges $\{c_{0,1}, c_{0,2}\}$ and $\{c_{0,2}, c_{0,3}\}$. Let H'_1 be the graph induced by $\cup_{i=1}^{q-1} E(c_i) \cup E(c'_0)$ and let H'_2 be the graph induced by $(E(H_1) \cup E(H_2)) \setminus E(H'_1)$. Then every vertex has degree 2 in H'_1 and H'_2 except $c_{0,1}$ and $c_{0,3}$ both of which have degree 1 in both H'_1 and H'_2.

Then $(\mathbb{Z}_{n-2} \cup \{\{\infty_j\} \mid 1 \leq j \leq 2\}, B_1 \cup B_2 \cup \{\{\infty_i, a_i, b_i\} \mid \{a_i, b_i\} \in E(H'_i)\} \cup \{\{\infty_1, \infty_2, c_{0,l}\} \mid l \in \{1, 3\}\})$ is the required decomposition.

Case 8: Suppose $n \equiv 1$ or $3 \pmod 6$, $\alpha = 0$, and that Q has a cycle of length at least 4.

A K_3-decomposition of $2K_n$ will be constructed on the vertex set $\mathbb{Z}_{n-2} \cup \{\infty_1, \infty_2\}$ such that the neighborhood of the vertex ∞_1 is Q.

Let $Q = \{c_0, ..., c_{q-1}\}$ where for each $i \in \mathbb{Z}_q$, the cycle $c_i = (c_{i,1}, \ldots, c_{i,l_i})$ has length l_i, $c_{j,k} \in \mathbb{Z}_{n-2}$ unless $(j, k) = (0, 2)$, $c_{0,2} = \infty_2$, and $l_0 \geq 4$. Since $n - 2 \equiv 1$ or $5 \pmod 6$, by Theorem 4.3, there exists a decomposition of K_{n-2} into triples and a near-Hamilton cycle. So let $(\mathbb{Z}_{n-2}, B_1)$ be a K_3-decomposition of $K_{n-2} - E(H_1)$, where H_1 is a near-Hamilton cycle named so that $H_1 = (c_{0,3}, c_{0,4}, c_{0,5}, \ldots, c_{0,l_0}, c_{1,1}, \ldots, c_{1,l_1}, \ldots, c_{q-1,1}, \ldots, c_{q-1,l_{q-1}})$ (so $c_{0,1}$ is omitted). Let $(\mathbb{Z}_{n-2}, B_2)$ be a K_3-decomposition of $K_{n-2} - E(H_2)$, where H_2 is the near-Hamilton cycle $H_2 = (c_{0,1}, c_{0,l_0}, c_{1,1}, c_{1,l_1}, \ldots, c_{q-1,1}, c_{q-1,l_{q-1}}, v_1, \ldots, v_{(n-2)-1-2q})$ where $v_1, \ldots, v_{(n-2)-1-2q}$ omit $c_{0,3}$ and are otherwise arbitrarily named. (Note that $c_{0,3} \neq c_{0,l_0}$ since $l_0 \geq 4$.) Then $c_{0,1}$ has degree 2 in H_2 and degree 0 in H_1, $c_{0,3}$ has degree 2 in H_1 and degree 0 in H_1, and every other vertex in $\mathbb{Z}_{n-2}$ has degree 2 in both H_1 and H_2.

Let H'_1 be the graph induced by $(E(H_1) \cup \{\{c_{i,1}, c_{i,l_i}\} \mid i \in \mathbb{Z}_q\}) \setminus (\{\{c_{i,l_i}, c_{i+1,1}\} \mid i \in \mathbb{Z}_q\} \cup \{\{c_{0,3}, c_{q-1,l_{q-1}}\}\})$ reducing the sum in the subscript modulo q. (Note that $\{c_{0,1}, c_{q-1,l_{q-1}}\} \notin E(H_1)$ so this edge is not removed). Note that in H'_1 every vertex in $\mathbb{Z}_{n-2}$ has degree 2 except $c_{0,1}$ and $c_{0,3}$, both of which have degree 1. Let H'_2 be the graph induced by $(E(H_1) \cup E(H_2)) \setminus E(H'_1)$. Note that in H'_2, every vertex in $\mathbb{Z}_{n-2}$ has degree 2 except $c_{0,1}$ and $c_{0,3}$, both of which have degree 1. The set of cycles formed by the components in H'_1 contains the cycles $c_1, ..., c_{q-1}$ and c'_0 where c'_0 is formed from c_0 by deleting the edges $\{c_{0,1}, c_{0,2}\}$ and $\{c_{0,2}, c_{0,3}\}$.

Then $(\mathbb{Z}_n \cup \{\infty_1, \infty_2\}, B_1 \cup B_2 \cup \{\{\infty_i, a_i, b_i\} \mid \{a_i, b_i\} \in E(H_i'), 1 \le i \le 2\} \cup \{\{\infty_1, \infty_2, c_{0,l}\} \mid l \in \{1, 3\}\})$ is the required K_3-decomposition.

Case 9: Suppose $n \equiv 1 \pmod 3$, $\alpha = 0$, and that Q has a 3-cycle.

A maximum packing is constructed on the vertex set $\mathbb{Z}_{n-4} \cup \{\{\infty_j\} \mid 1 \le j \le 4\}$ where the neighborhood of ∞_1 is $Q = \{c_0, \ldots, c_{q-1}\}$, where $c_0 = (\infty_2, \infty_3, \infty_4)$, and the $q-1$ other-cycles are defined on the vertex set $\mathbb{Z}_{n-4}$.

When $n = 4$, the result is trivial, and Q cannot have a 3-cycle when $n = 1$ (obviously) or when $n = 7$ (since then $Q = C_3 \cup C_3$, and $(n, Q) \ne (7, C_3 \cup C_3)$ by assumption). So it can be assumed that $n \ge 10$. Then $w = n - 4 \equiv 0 \pmod 3$ and $4 \le w - 1$. So by Lemma 4.4, there exists a K_3-decomposition $(\mathbb{Z}_{n-4}, B)$ of $2K_{n-4} - (E(H_1) \cup E(H_2) \cup E(H_3) \cup E(H_4))$ where H_1, H_2, H_3, and H_4 are 2-regular graphs and $H_1 \equiv Q \setminus c_0$.

Let $(\{\{\infty_j\} \mid 1 \le j \le 4\}, B_3)$ be a K_3-decomposition of $2K_4$ (where the neighborhood of ∞_1 is the 3-cycle $(\infty_2, \infty_3, \infty_4)$).

Then $(\mathbb{Z}_{n-4} \cup \{\{\infty_j\} \mid 1 \le j \le 4\}, B \cup B_3 \cup \{\{\infty_i, a_i, b_i\} \mid \{a_i, b_i\} \in E(H_i), 1 \le j \le 4\}$ is the required decomposition.

Case 10: Suppose $n \equiv 4 \pmod 6$, $\alpha = 0$, and that Q has a cycle of length at least 5.

A maximum packing is constructed on the vertex set $\mathbb{Z}_{n-2} \cup \{\{\infty_j\} \mid 1 \le j \le 2\}$ where the neighborhood of ∞_1 is $Q = \{c_0, \ldots, c_{q-1}\}$.

For each $i \in \mathbb{Z}_q$, let $c_i = (c_{i,1}, \ldots, c_{i,l_i})$ where l_i is the length of c_i, $l_0 \ge 5$, $c_{j,k} \in \mathbb{Z}_{n-2}$ for all $(j,k) \ne (0,2)$, and $c_{0,2} = \infty_2$. In this case, $n - 2 \equiv 2 \pmod 6$ and thus $\binom{(n-2)}{2} - \frac{n-2}{2} - (n-2-2)$ and $\binom{(n-2)}{2} - \frac{n-2}{2}$ are both divisible by 3, and since $n - 2 \ge 8$, both quantities are nonnegative. So by Theorem 4.3, let $(\mathbb{Z}_{n-2}, B_1)$ be a K_3-decomposition of $K_{n-2} - (E(H_1) \cup E(F_1))$ and let $(\mathbb{Z}_{n-2}, B_2)$ be a K_3-decomposition of $K_{n-2} - (E(F_2))$ where H_1 is a $n-4$ cycle, F_1 and F_2 are 1-factors, and H_1 and F_2 are named as follows: Let $H_1 = (c_{0,4}, c_{0,5}, \ldots, c_{0,l_0}, c_{1,1}, \ldots, c_{1,l_1}, c_{2,1}, \ldots, c_{2,l_2}, \ldots, c_{q-1,1}, \ldots, c_{q-1,l_{q-1}})$ (so that $c_{0,1}$ and $c_{0,3}$ are omitted). Let F_2 be defined to contain the edges $\{c_{i,1}, c_{i,l_i}\}$ for each $i \in \mathbb{Z}_q$ as well as the edge $\{c_{0,3}, c_{0,4}\}$ and finally $\frac{n-2-2q-2}{2}$ arbitrarily named edges. (Note that $c_{0,4} \neq c_{0,l_0}$ since $l_0 \geq 5$.)

Note that $\cup_{i=1}^{q-1} E(c_i) \cup E(c'_0) \subset E(H_1) \cup E(H_2)$ where c'_0 is formed from c_0 by removing the edges $\{c_{0,1}, c_{0,2}\}$ and $\{c_{0,2}, c_{0,3}\}$. Let H'_1 be the graph induced by $\cup_{i=1}^{q-1} E(c_i) \cup E(c'_0)$ and let H'_2 be the graph induced by $(E(H_1) \cup E(H_2)) \setminus E(H'_1)$. Then every vertex has degree 2 in H'_1 and H'_2 except $c_{0,1}$ and $c_{0,3}$ both of which have degree 1 in both H'_1 and H'_2.

Then $(\mathbb{Z}_{n-2} \cup \{\{\infty_j\} \mid 1 \leq j \leq 2\}, B_1 \cup B_2 \cup \{\{\infty_i, a_i, b_i\} \mid \{a_i, b_i\} \in E(H'_i)\} \cup \{\{\infty_1, \infty_2, c_{0,l}\} \mid l \in \{1, 3\}\})$ is the required decomposition.

Note that this covers all the cases since Q must contain an odd cycle when $n \equiv 2 \pmod 6$ and $\alpha = 0$, Q cannot contain all 3-cycles when $n \equiv 0 \pmod 3$ ($|Q| = 2 \pmod 3$), and Q cannot consist entirely of even cycles when $n \equiv 4 \pmod 6$. $\square$

Chapter 5

Quadratic Excesses or Paddings of Covers with Triples of λK_n

5.1 Introduction

Having found a solution to the quadratic leave problem (completed in Chapter 3), this chapter will focus on the quadratic excess problem. As mentioned in the introduction, Colbourn and Rosa found necessary and sufficient conditions for a quadratic graph to be the excess of a cover of λK_n when $\lambda = 1$ (see [10]). In this chapter, their results are extended to all λ (see Theorem 5.2).

5.2 Results

In this section, the main result of the chapter is given, namely necessary and sufficient conditions for a quadratic graph to be the excess of a cover of λK_n (naturally it is assumed that $\lambda \geq 1$ in this chapter). However, to begin a very well known theorem on maximum packings and minimum covers of λK_n (packings and covers for which the leave and excess have as few edges as possible) is given.

Theorem 5.1. *[14, 23] Let $\lambda \geq 1$ and $n \neq 2$. Let P (or L) be any multigraph with the least number of edges in which all vertices have degree congruent to $\lambda(n-1)$ (mod 2) and with $|E(P)| + \frac{\lambda n(n-1)}{2} \equiv 0$ (mod 3) (or $\frac{\lambda n(n-1)}{2} - |E(L)| \equiv 0$ (mod 3) respectively). Then there exists a K_3-decomposition of $\lambda K_n \cup E(P)$ (or $\lambda K_n - E(L)$ respectively).*

The statement and proof of the main theorem are now given.

Theorem 5.2. *Let Q be a quadratic graph on n vertices. Then Q is the excess of a cover of λK_n if and only if*

1. $\lambda(n-1)$ *is even,*

2. $|E(Q)| + |E(\lambda K_n)| \equiv 0$ *(mod 3), and*

3. $n \neq 2$.

Proof. The necessity of Conditions (1) and (2) follows since each vertex in each triple has even degree and each triple contains 3 edges respectively. The necessity of Condition (3) is clear since $\lambda K_2 + E(Q)$ contains no copies of K_3.

To prove the sufficiency, suppose that $(1-3)$ hold. Several cases are considered in turn.

Case 1: $n \equiv 1, 3 \pmod 6$

By Theorem 5.1, let (V, B_1) be a K_3-decomposition of $(\lambda - 1)K_n$ (with $L = \emptyset$). By Condition (2) and Theorem 1.3, let (V, B_2) be a cover of K_n with excess Q. Then $(V, B_1 \cup B_2)$ is the required cover.

Case 2: $n \equiv 5 \pmod 6$

Let $\epsilon \in \{1, 2, 3\}$ with $\epsilon \equiv \lambda \pmod 3$. Note that $|E(\epsilon K_n)| \equiv |E(\lambda K_n)| \pmod 3$. Since $n \equiv 5 \pmod 6$, by Theorem 5.1, there exists a K_3-decomposition (V, B_1) of $(\lambda - \epsilon)K_n$.

If $\epsilon = 1$ then by Condition (2) and Theorem 1.3, let (V, B_2) be a cover of K_n with excess Q.

Suppose $\epsilon = 2$. By Condition (2), $|E(Q)| \equiv 1 \pmod 3$. Since $n \equiv 2 \pmod 3$, some vertex in V has degree 0 in Q, say x. Let $c = (c_0, c_1, ..., c_i)$ be a cycle in Q and let $c' = (c_0, x, c_1, ..., c_i)$. Form Q' from Q by replacing c with c'. Q' is quadratic on the vertex set V and $|E(Q')| \equiv 2 \pmod 3$, so by Theorem 1.3, let (V, B_2') be a cover of K_n with excess Q'. By Theorem 5.1, let (V, B_2'') be a maximum packing of K_n with leave the 4-cycle (c_0, x, c_1, d) where d is any vertex in V other than c, x_0, and x_1 (this exists since in this case $n \equiv 5 \pmod 6$ so $n \geq 5$). Let $B_2 = B_2' \cup B_2'' \cup \{\{c_0, c_1, d\}\}$.

If $\epsilon = 3$ then by Condition 2 $|E(Q)| \equiv 0 \pmod 3$. Since $n \equiv 2 \pmod 3$, some vertex in V has degree 0 in Q, say x. Let $c = (c_0, c_1, ..., c_i) \in Q$ and let $c' = (c_0, x, c_1, ..., c_i)$. Form Q' from Q by replacing c with c'. Q' is quadratic on the vertex set V and $|E(Q')| \equiv 1 \pmod 3$, so by the previous case in this proof when $\epsilon = 2$, let (V, B_2') be a cover of $2K_n$ with excess Q'. By Theorem 5.1, let (V, B_2'') be a maximum packing of K_n with leave the 4-cycle (c_0, x, c_1, d) where d is any vertex in V other than x, c_0, and c_1. Let $B_2 = B_2' \cup B_2'' \cup \{\{c_0, c_1, d\}\}$.

Then in each subcase ($\epsilon = 1, 2$, and 3), $(V, B_1 \cup B_2)$ is the required cover.

Case 3: $n \equiv 4 \pmod 6$

Since $n \equiv 4 \pmod 6$, by Condition (1), λ is even. By Theorem 5.1, let (V, B_1) be a K_3-decomposition of $(\lambda - 2)K_n$.

By Condition (2), $|E(Q)| \equiv 0 \pmod 3$. Hence, Q also satisfies the conditions for an excess for $v = n - 1$ and $\lambda = 1$. So let $x \in V$ be an isolated vertex in Q (in this case $|V(Q)| \equiv 1 \pmod 3$ and $|E(Q)| \equiv 0 \pmod 3$ so such an x exists) and let $(V \setminus \{x\}, B_2)$ be a cover of K_{n-1} with excess Q. By Theorem 5.1, let (V, B_3) be a maximum packing of K_n with leave Q' consisting of a $K_{1,3}$ and $\frac{n-4}{2}$ independent edges, where the vertex set of the $K_{1,3}$ is $\{w, x, y, z\} \subset V$ with y being the vertex of degree 3. Then $(V, B_1 \cup B_2 \cup B_3 \cup \{\{x, a_i, b_i\} \mid \{a_i, b_i\}$ is an independent edge in $Q'\} \cup \{\{x, y, z\}, \{x, y, w\}\})$ is the required decomposition.

Case 4: $\lambda = 2$ and $n \equiv 0, 2 \pmod 6$

First suppose Q consists entirely of 2-cycles and isolated vertices. Hence $|E(Q)|$ is even, and by Condition (2), $|E(Q)| \equiv 0$ or $1 \pmod 3$ when $n \equiv 0$ or $2 \pmod 6$ respectively (recall $\lambda = 2$ in this case). Hence $|E(Q)| \equiv 0$ or $4 \pmod 6$ when $n \equiv 0$ or $2 \pmod 6$ respectively and thus the number of isolated vertices in Q is equivalent to 0 or 4 (mod 6) when $n \equiv 0$ or 2 (mod 6) respectively. If Q contains no isolated vertices (so $n \equiv 0 \pmod 6$), then by Theorem 5.1, for each $k \in \{1, 2\}$ let (V, B_k) be a minimum cover of K_n with the same 1-factor excess. Then $(V, B_1 \cup B_2)$ is the required cover. Otherwise Q contains at least 4 isolated vertices, and this case is handled below by using the observation in the next paragraph.

Note that if Q is a quadratic graph in which there are three isolated vertices, say a, b, and c, in Q, then Q is a quadratic excess if and only if $Q \cup \{\{a, b\}, \{a, c\}, \{b, c\}\}$ is a quadratic excess. If Q contains at least 3 isolated vertices, then add a cycle of length 3 on three of the isolated vertices to Q.

In light of the last two paragraphs, to complete the proof of Case 4 it now suffices to consider the situation where Q has a cycle $c = (v_0, v_1, ..., v_x)$ of length $x + 1 \geq 3$. Form Q' from Q by replacing c with $c' = (v_1, v_2, ..., v_x)$. Note that $|E(Q')| \equiv 2 \pmod 3$ and $0 \pmod 3$ when $n \equiv 0$ and $2 \pmod 6$ respectively. Further note that $v_0 \notin V(Q')$ and that Q' satisfies the conditions for an excess when $\lambda = 1$ and $n' = n - 1 \equiv 5 \pmod 6$ and for an excess when $\lambda = 1$ and $n' = n - 1 \equiv 1 \pmod 6$. So by Theorem 1.3, let $(V \setminus \{v_0\}, B_1)$ be a cover of K_{n-1} with excess Q'.

By Theorem 5.1, let (V, B_2) be a maximum packing of K_n with leave a 1-factor F named to contain the edges $\{v_1, v_x\}$ and $\{v_0, d\}$ where d is a vertex for which $\{v_1, v_x, d\} \in B_1$. Then $(V, (B_1 \cup B_2 \setminus \{\{v_1, v_x, d\}\}) \cup \{\{v_0, a_i, b_i\} \mid \{a_i, b_i\} \in F \setminus \{\{v_1, v_x\}, \{v_0, d\}\}\} \cup \{\{v_0, v_1, d\}, \{v_0, v_1, v_x\}, \{v_0, v_x, d\}\})$ is the required cover.

Case 5: $\lambda > 2$ and $n \equiv 0 \pmod 6$

By Condition (1), λ is even. Hence, by Theorem 5.1, let (V, B_1) be a K_3-decomposition of $(\lambda - 2)K_n$. By Condition (2), $|E(Q)| \equiv 0 \pmod 3$. Hence, by Case 4, let (V, B_2) be a cover of $2K_n$ with excess Q. Then $(V, B_1 \cup B_2)$ is the required cover.

Case 6: $\lambda > 2$ and $n \equiv 2 \pmod 6$

By Condition 1, λ is even. Let $\epsilon \in \{2, 4, 6\}$ with $\epsilon \equiv \lambda \pmod 6$. Note that $|E(\lambda K_n)| \equiv |E(\epsilon K_n)| \pmod 3$ and $\epsilon \equiv \lambda \pmod 2$ so Q satisfies the necessary conditions for an excess of λK_n precisely when it satisfies the necessary conditions for an excess of ϵK_n. By Condition (3), $n \geq 8$, so by Theorem 5.1, let (V, B_1) be a K_3-decomposition of $(\lambda - \epsilon)K_n$.

If $\epsilon = 2$, by Case 4, let (V, B_2) be a cover of $2K_n$ with excess Q.

If $\epsilon = 4$ then by Condition 2 $|E(Q)| \equiv 2 \pmod 3$. First suppose Q consists only of 2-cycles and isolated vertices. Since $|E(Q)| \equiv 4 \equiv n \pmod 6$, the number of isolated vertices in Q must be a multiple of 6. If Q has no isolated vertices, then let Q' consist of two of the 2-cycles and Q'' consist of the remaining $\frac{n}{2} - 2 \geq 2$ 2-cycles ($n \geq 8$ in this case). Note that $|E(Q')| \equiv |E(Q'')| \equiv 1 \pmod 3$ so by Case 4, let (V, B_2') be a cover of $2K_n$ with excess Q'

and (V, B_2'') be a cover of $2K_n$ with excess Q''. Let $B_2 = B_2' \cup B_2''$. Otherwise Q has at least 6 isolated vertices; add a 3-cycle on three of the isolated vertices to Q.

It remains to consider the case where Q has a cycle $c = (v_0, v_1, ..., v_x)$ of length $x+1 \geq 3$. Form Q' from Q by replacing c with $c' = (v_1, v_2, ..., v_x)$. Note $|E(Q')| \equiv 1 \pmod 3$ so by Case 4, let (V, B_2') be a cover of $2K_n$ with excess Q'. By Theorem 5.1, let (V, B_2'') be a packing of $2K_n$ with leave $\{\{c_1, c_x\}, \{c_1, c_x\}\}$. Let $B_2 = B_2' \cup B_2'' \cup \{\{c_0, c_1, c_x\}\}$.

If $\epsilon = 6$ then $|E(Q)| \equiv 0 \pmod 3$. Since $n \equiv 2 \pmod 6$, there are at least two vertices, say v_0 and v_1 such that $v_0, v_1 \notin V(Q)$. Form Q' from Q by adding the 2-cycle (v_0, v_1). Note $|E(Q')| \equiv 2 \pmod 3$, so by the previous case where $\epsilon = 4$, let (V, B_2') be a cover of $4K_n$ with excess Q'. By Theorem 5.1, let (V, B_2'') be a packing of $2K_n$ with leave $\{\{v_0, v_1\}, \{v_0, v_1\}\}$. Let $B_2 = B_2' \cup B_2''$.

In each case, $(V, B_1 \cup B_2)$ is the required cover, so the result is proved. $\square$

Chapter 6

Conclusion

To conclude this dissertation, it seems appropriate to mention some applications of the work done as well as some future directions of research. Some of the mathematical applications of the results in this dissertation (such as telling when two 2-fold triple systems are isomorphic) were already discussed, so at this point, a couple of applications of the general topic of graph decompositions and structure within them are mentioned in relation to other topics. One useful application of graph decompositions involves scheduling problems. For instance, a 1-factorization of K_{12} would correspond to an 11 week football schedule in which each team plays each other team exactly once. A structure question related to this application is whether the schedule could be made so that six fixed games, say rivalry games, appear on the last week of the season. (Incidentally, this can be done.) In terms of more scientific applications, there is the following application. In research on viruses, a method known as the Ouchterlony method tests how antigens interact. Around the edge of a Petri dish, v of n antigens are placed where they can diffuse, and then the interaction of neighboring antigens is observed. For research purposes, it may be beneficial to have each pair of antigens appear as neighbors on exactly λ Petri dishes. This can be modeled mathematically as a v-cycle-decomposition of λK_n. In terms of the research in this dissertation, the topics covered in Chapter 2 directly correspond to a slight variant of the Ouchterlony method. Suppose now that there are $n + m$ antigens which are to be placed into two groups, one of size m and one of size n (for instance one group may share a specific trait while the other group may share a different trait). It may be desirable to see how two antigens in the same group interact λ_1 times while only seeing how antigens in different groups interact λ_2 times. This corresponds to a decomposition of the graph $\lambda_1 K_n \vee_{\lambda_2} \lambda_1 K_m$ that was discussed in Chapter 2.

In terms of future research, several problems seem interesting. An original topic for this dissertation involved finding necessary and sufficient conditions for a gregarious K_3-decomposition of $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$, where in this setting gregarious simply means that each triple contains two mixed edges and one pure edge. In [13], El-Zanati, Punnim, and Rodger solved this problem when $\lambda_1 = 1$ and $\lambda_2 = 2$. Little other work has been done on the subject, although some minor results have been obtained. This would be an interesting problem to consider, as would the problem of finding necessary and sufficient conditions for a K_3-decomposition (not necessarily gregarious) of $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$ when $\lambda_1 < \lambda_2$; both problems seem difficult.

Bibliography

[1] A. Bialostocki and J. Schönheim, Packing and Covering of the Complete Graph with 4-cycles, *Canadian Mathematical Bulletin* 18 (5) (1975), 703-708.

[2] R.C. Bose and T. Shimamoto, Classification and analysis of partially balanced incomplete block designs with two associate classes, *Journal of the American Statistical Association* 47 (1952) 151 – 184.

[3] D. Bryant, Hamilton cycle rich two-factorizations of complete graphs, *Journal of Combinatorial Designs* 12 (2) (2004), 147 – 155.

[4] D. Bryant, D. Horsley and B. Maenhaut, Decompositions into 2-regular subgraphs and equitable partial cycle decompositions, *Journal of Combinatorial Theory. Series B* 93 (1) (2005), 67 – 72.

[5] D. Bryant, D. Horsley, and W. Pettersson, Cycle decompositions V: Complete graphs into cycles of arbitrary lengths, submitted.

[6] D. Bryant and D. Horsley, An asymptotic solution to the cycle decomposition problem for complete graphs, *Journal of Combinatorial Theory. Series A* 117 (8) (2010), 1258 – 1284.

[7] J. Chaffee and C.A. Rodger, Neighborhoods in Maximum Packings of $2K_n$ and Quadratic Leaves of Triple Systems, *Jounral of Combinatorial Designs*, to appear.

[8] C.J. Colbourn, D.G. Hoffman and R. Rees, A new class of group divisible designs with block size three, *Journal of Combinatorial Theory Series A* 59 (1) (1992) 73 – 89.

[9] C.J. Colbourn and A. Rosa, Element neighbourhoods in twofold triple systems, *Journal of Geometry* 30 (1) (1987), 36 – 41.

[10] C.J. Colbourn and A. Rosa, Quadratic excesses of coverings by triples, *Ars Combinatoria* 24 (1987), 23-30.

[11] C.J. Colbourn and A. Rosa, Quadratic leaves of maximal partial triple systems, *Graphs and Combinatorics* 2 (4) (1986), 317-337.

[12] C.J. Colbourn, A. Rosa, and S. Znám, The spectrum of maximal partial Steiner triple systems, *Designs, Codes and Cryptography. An International Journal* 3 (3), (1993), 209 – 219.

[13] S.I. El-Zanati, Narong Punnim, C.A. Rodger, Gregarious GDDs with two associate classes, *Graphs and Combinatorics*, 26 (6) (2010), 775-780.

[14] M.K. Fort Jr. and G.A. Hedlund, Minimal coverings of pairs by triples, *Pacific Journal of Mathematics* 8 (1958), $709-719$.

[15] H.L. Fu and C.A. Rodger, Group divisible designs with two associate classes: $n = 2$ or $m = 2$, *Journal of Combinatorial Theory A* 83 (1) (1998) $94-117$.

[16] H.L. Fu, C.A. Rodger and D.G. Sarvate, The existence of group divisible designs with first and second associates, having block size 3, *Ars Combinatoria* 54 (2000) $33-50$.

[17] Haim Hanani, The existence and construction of balanced incomplete block designs, *Annals of Mathematical Statistics*, 32 (1961), 361-386.

[18] D.G. Hoffman, C.A. Rodger, and A. Rosa, Maximal sets of 2-factors and Hamiltonian cycles, *Journal of Combinatorial Theory. Series B* 57 (1) (1993), $69-76$.

[19] Thomas P. Kirkman, On a problem in combinations, *Cambridge and Dublin Math*, J 2 (1847), 191-204.

[20] H. Lenz and G. Stern, Steiner triple systems with given subspaces; another proof of the Doyen-Wilson-Theorem, *Unione Matematica Italiana. Bollettino. A* (5), 17 (1) (1980) $109-114$.

[21] C.C. Lindner and C.A. Rodger, Design theory, *CRC Press*, Boca Raton, FL, 2009.

[22] L. McCauley and C.A. Rodger, Hamilton cycle rich 2-factorizations of complete multipartite graphs, Graphs and Combinatorics 24 (1) (2008), $47-52$.

[23] E. Mendelsohn, N. Shalaby and H. Shen, Nuclear designs, *Ars Combinatoria* 32 (1991), 225-238.

[24] S. Milici, Minimum coverings of the complete graph with 5-cycles, *Journal of Combinatorial Mathematics and Combinatorial Computing*, 57 (2006), 33-46

[25] N. Pabhapote and N. Punnim, Group divisible designs with two associate classes and $\lambda_2 = 1$, *International Journal of Mathematics and Mathematical Sciences* (2011).

[26] J. Petersen, Die Theorie der regulären graphs, *Acta Mathematica* 15 (1) (1891), $193-220$.

[27] C.A. Rodger and S.J. Stubbs, Embedding partial triple systems, *Journal of Combinatorial Theory Series A* 44 (2), (1987) $241-252$.

[28] E.A. Severn, Maximal Partial Triple Systems, *Computer Science Technical Report* 172/84, University of Toronto (1984).

[29] J.E. Simpson, Langford sequences: perfect and hooked,*Discrete Math* 44 (1) (1983) $97-104$.